THE UNIVERSITY OF ALBERTA

THE APPROXIMATE DISTRIBUTION OF THE RATIO OF THE SQUARE SUCCESSIVE DIFFERENCE TO THE SQUARE DIFFERENCE IN OBSERVATIONS FROM A STATIONARY NORMAL MARKOV PROCESS

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "The Approximate Distribution of the Ratio of the Square Successive Difference to the Square Difference in Observations from a Stationary Normal Markov Process", submitted by Adery C. A. Patton, B.Sc. in partial fulfilment of the requirements for the degree of Master of Science.

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ABSTRACT

The purpose of this thesis is to obtain the approximate distribution of the ratio of the square successive difference to the square difference in observations from a stationary normal Markov process. The relevant literature is reviewed in Chapter I. Chapter II is concerned with Daniels' development of the distributions of a sample serial correlation coefficient for the linear Markov process with known and unknown means. In Chapter III, following Daniels' method, we obtain the approximate distribution of a slightly different sample serial correlation coefficient. Chapter IV deals with the distribution of the ratio of the square successive difference to the square difference which is obtained from the results of the previous chapter. Also in Chapter IV, we compare the moments of this ratio for the case where $\rho = 0$ with those obtained by Williams.

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CHAPTER I

INTRODUCTION

When analysing data that possess trends, these trends are usually removed and the remaining effects are studied. For this purpose the method of successive differences has been employed. However, the effect of trend on dispersion must be considered in addition to determining if a trend actually exists.

The usefulness of the method of successive differences was first realized by ballisticians who were concerned with the problem of minimizing several effects in measuring the dispersion of the distance travelled by a shell. The first to use this method was Vallier (14). In later papers this and related problems were discussed.

Von Neumann et al (9) mentioned that the mean square successive difference δ^2, where

$$\delta^2 = \frac{\sum_{i=1}^{n-1} (x_{i+1} - x_i)^2}{n - 1} ,$$

may be employed to estimate the standard deviation so as to minimize the effects of the trend on dispersion. Also, by testing δ^2 against

$$s^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n} ,$$

which measures the variance disregarding the order of the observations and hence including any trend effects present, it may be determined if a trend actually exists or not. In this paper they discussed the distribution of δ^2. They assumed that the samples were drawn from a normal population.

In 2 later papers, von Neumann (10),(11) studied the ratio

$$\eta = \frac{\delta^2}{s^2} .$$

Here, as before, the samples were taken from a normal population. He discussed a more general distribution of which that of η was a special case.

J. D. Williams (16) obtained the moments of η by using generating functions. Like von Neumann and others, he assumed the samples to consist of independent, normal random variables, $x_1, \ldots, x_n$.

The stochastic process,

$$x_s = \rho\, x_{s-1} + z_s , \quad (s = 1, 2, \ldots, n) , \quad |\rho| < 1 ,$$

where ρ is the serial correlation coefficient, has been discussed by Koopmans (5). The $\{z_s\}$ are assumed to be independent, $N(0, \sigma^2)$ random variables. As noted by Dixon (3), Koopmans has shown that it is sufficient to know the distribution of

$$\frac{\sum_{i=1}^{n-1} (x_i - \bar{x})(x_{i+1} - \bar{x})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

to test the hypothesis that $\rho = 0$.

Several results on successive difference estimation including significance tests were obtained by Tintner (13). In one of these tests, selected sets of differences were used.

Morse and Grubbs (8) considered the estimation of variance by the use of successive differences of order greater than one. They felt that this method would be valuable in cases where the type of trend is unknown.

It was noted by Morse and Grubbs that the theory of successive differences and the problem of serial correlation are related. Dixon [(3), page 119] states, "The development of the distribution theory and significance criteria [of serial correlation coefficients] was retarded by the fact that the successive differences or successive products of statistical variates are not independent." A lot of these difficulties have been overcome and we are able to study the distributions of serial correlation coefficients and the theory of successive differences more extensively.

A circular definition of the serial correlation coefficient was suggested by Hotelling. By using such modified statistics, further progress in developing the distribution theory was made by others such as R. L. Anderson, Dixon, Koopmans and Madow. As noted by Daniels [(2), page 169], "The exact distribution is known for the circular coefficient of any lag from an uncorrelated normal process and, more generally, from a circularly modified normal process of autoregressive type. Quenouille (1949) obtained by

the same method the exact joint distribution of circular coefficients of different lags."

R. L. Anderson (1) obtained the exact distribution of $\bar{r}$ for the null case $(\rho = 0)$, where

$$\bar{r} = \frac{x_1 x_2 + \dots + x_T x_1}{x_1^2 + \dots + x_T^2}$$

is an estimate of the serial correlation coefficient, ρ, in a circular universe. Since the exact distribution is complicated, Koopmans (5) found a definite integral and Dixon (3) an explicit expression for an approximate distribution of $\bar{r}$ for $\rho = 0$. These results were shown to be equivalent by Rubin (12). Madow (7), using results of this work, extended Anderson's results to obtain the exact distribution of the serial correlation coefficient for the non-null circular model. Following Madow's method, an approximate distribution of the sample serial correlation coefficient for the case $\rho \neq 0$ was found by Leipnik (6). Daniels (2) employed the method of steepest descent to obtain the approximate distribution of the sample serial correlation coefficient for the circular and non-circular Markov processes with known and unknown means.

In this thesis we will consider the case of the non-circular Markov process with known and unknown means. We disregard the circular case for it is felt that it is an artificial means of studying a time series. We review Daniels' method and work in

obtaining the approximate distribution of the sample serial correlation coefficient

$$r = \frac{c}{c_0} ,$$

where

$$c = x_1 x_2 + \ldots + x_{n-1} x_n ,$$

$$c_0 = \tfrac{1}{2} x_1^2 + x_2^2 + \ldots + x_{n-1}^2 + \tfrac{1}{2} x_n^2 ,$$

for the non-circular Markov process. We then use Daniels' method to obtain the approximate distribution of a slightly different sample serial correlation coefficient which is defined as follows:

$$r = \frac{c}{c_0} ,$$

where

$$c = \tfrac{1}{2} x_1^2 + x_1 x_2 + \ldots + x_{n-1} x_n + \tfrac{1}{2} x_n^2 ,$$

$$c_0 = x_1^2 + \ldots + x_n^2 .$$

This is for the non-circular Markov process.

From the approximate distributions of r, as defined in the latter case, we obtain the approximate distributions of

$$D = \frac{\sum_{i=1}^{n-1} (x_{i+1} - x_i)^2}{\sum_{i=1}^{n} x_i^2}$$

and

$$D_1 = \frac{\sum_{i=1}^{n-1} (x_{i+1} - x_i)^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2} .$$

This is related to the work of von Neumann (10), (11) and Williams (16) since D_1 is similar to η for large values of n except that we assume the sample $x_1, \ldots, x_n$ to consist of serially correlated rather than independent, normal random variables. By putting the serial correlation coefficient equal to zero $(\rho = 0)$, our distribution is approximately the same as that discussed by von Neumann and Williams for large values of n. The moments m_0 and m_1 of

$$\frac{A}{B} = \frac{\sum_{j=1}^{n-1} (X_{j+1} - X_j)^2}{\sum_{j=1}^{n} (X_j - \bar{X})^2} ,$$

where $\{X_1, \ldots, X_n\}$ is a sample from a normal population with mean zero and variance σ^2, obtained by Williams are the same as those obtained by using our approximate distribution of D_1 with $\rho = 0$. The higher moments obtained using the probability density of D_1 with $\rho = 0$ are approximately equal for large n to those of $\frac{A}{B}$ obtained by Williams.

CHAPTER II

THE APPROXIMATE DISTRIBUTIONS OF AN ESTIMATE OF THE SERIAL CORRELATION COEFFICIENT FOR A NON-CIRCULAR MARKOV PROCESS WITH KNOWN AND UNKNOWN MEANS AS DEVELOPED BY H. E. DANIELS

1. Linear Markov Processes

Consider a subset of time points

$$t_1 < t_2 < \dots < t_n$$

and the random variables $X(t_j)$ at time t_j. The joint distribution of $X(t_1), \dots, X(t_n)$ is denoted by $p[X(t_1), \dots, X(t_n)]$. We have a completely stationary process if

$$p[X(t_1), \dots, X(t_n)] = p[X(t_1 - \tau'), \dots, X(t_n - \tau')]$$

for all subsets $t_1 < t_2 < \dots < t_n$. That is, the joint distribution is invariant under a change of time origin. A process is stationary of order k if

$$\mathcal{E}[X^{\alpha_1}(t_1) \dots X^{\alpha_n}(t_n)] = \mathcal{E}[X^{\alpha_1}(t_1 - \tau') \dots X^{\alpha_n}(t_n - \tau')]$$

for all τ' and all $\alpha_1 + \dots + \alpha_n \leq k$.

We shall now consider the cases of order one and two. If the process is stationary of order one,

$$\mathcal{E}[X(t)] = \mathcal{E}[X(t - \tau')] = \mu$$

for all τ'. Thus the mean is independent of time. If the process is

stationary of order two, we have

$$\mathcal{E}\,[X^2(t)] = \mathcal{E}\,[X^2(t - \tau')]$$

for all τ' and

$$\mathcal{E}\,[X(t_1)\ X(t_2)] = \mathcal{E}[X(t_1-\tau')\ X(t_2-\tau')]$$

for all τ'. Then

$$\begin{aligned} Var[X(t)] &= \mathcal{E}\,[X^2(t)] - \{\mathcal{E}\,[X(t)]\}^2 \\ &= Var[X(t - \tau')] \\ &= \sigma^2 \end{aligned}$$

for all τ'. That is, the variance is independent of time. We call σ^2 the autovariance. Also,

$$\begin{aligned} Cov[X(t_1), X(t_2)] &= \mathcal{E}\,[X(t_1)\ X(t_2)] - \mathcal{E}\,[X(t_1)]\,\mathcal{E}[X(t_2)] \\ &= Cov[X(t_1-\tau'), X(t_2-\tau')] \\ &= C(t_2 - t_1) \end{aligned}$$

for all τ'. $C(t_2 - t_1)$ is called the autocovariance and, as shown, is independent of the time origin.

The autocorrelation function $\rho(\tau)$ is defined as:

$$\frac{C(t_2 - t_1)}{\sigma^2} = \rho(\tau) \quad ,$$

where $\tau = t_2 - t_1$ is the lag. $\rho(\tau)$ measures the degree to which variables at time difference τ are correlated. $\rho(\tau)$ is an even function of τ. Also, it can be shown that necessary and sufficient conditions for a function to be an autocorrelation function are that $\rho(0) = 1$ and $\rho(\tau)$ is positive semi-definite.

Let us consider the linear Markov process in discrete time defined by

$$x_s = \rho\, x_{s-1} + e_s$$

for all s, where the constant ρ is called the serial correlation coefficient and the $\{e_s\}$ are independently and identically distributed, $N(0,1)$ random variables. As shown below, such a process ultimately becomes stationary if $|\rho| < 1$.

If we assume that the initial state is x_0 then for $s = 1, 2, \ldots, n$

$$x_s = \rho^s x_0 + \rho^{s-1} e_1 + \rho^{s-2} e_2 + \ldots + \rho^2 e_{s-2} + \rho\, e_{s-1} + e_s \,.$$

Since $\mathcal{E}\,(e_s) = 0 \qquad (s = 1, 2, \ldots, n)$,

$$\mathcal{E}\,(x_s) = \rho^s\, x_0$$

and this approaches zero as s becomes large if $|\rho| < 1$. Then the process becomes stationary to the first order.

Since

$$\mathcal{E}\,(e_i^{\,2}) = \mathrm{Var}(e_i) = 1$$

and

$$\mathcal{E}\,(e_i\, e_j) = 0 \qquad (i \neq j) \quad ,$$

we have

$$\begin{aligned} \mathrm{Var}(x_s) &= \mathcal{E}(x^2_s) - [\mathcal{E}(x_s)]^2 \\ &= (\frac{1-\rho^{2s}}{1-\rho^2} + \rho^{2s}\, x_0^2) - (\rho^s\, x_0)^2 \\ &= \frac{1-\rho^{2s}}{1-\rho^2} \quad . \end{aligned}$$

If $|\rho| < 1$, $\mathrm{Var}(x_s)$ approaches $\frac{1}{1-\rho^2}$ as s becomes large. Now, for any fixed $\tau > 0$

$$\begin{aligned} \mathrm{Cov}(x_s, x_{s-\tau}) &= \mathcal{E}\,(x_s\, x_{s-\tau}) - \mathcal{E}(x_s)\,\mathcal{E}(x_{s-\tau}) \\ &= \left[\rho^\tau \left(\frac{1-\rho^{2(s-\tau)}}{1-\rho^2}\right) + \rho^{2s-\tau}\, x_0^2 \right] - (\rho^s x_0)(\rho^{s-\tau} x_0) \\ &= \rho^\tau \left(\frac{1-\rho^{2(s-\tau)}}{1-\rho^2}\right) \end{aligned}$$

and if $|\rho| < 1$

$$\mathrm{Cov}(x_s, x_{s-\tau}) \rightarrow \frac{\rho^\tau}{1-\rho^2} \qquad \text{as} \qquad s \rightarrow \infty \quad .$$

Also, for any fixed $\tau < 0$ $\quad \tau = -|\tau|$ and

$$\mathrm{Cov}(x_s, x_{s-\tau}) = \mathrm{Cov}(x_s, x_{s+|\tau|}) = \rho^{|\tau|} \left(\frac{1-\rho^{2s}}{1-\rho^2}\right)$$

and if $|\rho| < 1$

$$\mathrm{Cov}(x_s, x_{s-\tau}) \rightarrow \frac{\rho^{|\tau|}}{1-\rho^2} \quad \text{as} \quad s \rightarrow \infty \quad .$$

Hence, if $|\rho| < 1$ the process becomes stationary to the second order which, in the case of a normal process, implies complete stationarity. In many problems in which the process has been in existence for a very long time it is, therefore, realistic to assume stationarity.

2. Saddlepoint Approximation

In the present section we discuss section 2 of Daniels' paper. Here and in the subsequent sections of this chapter all references to Daniels are to his 1956 paper (2). The distribution of statistics of the form

$$r = \frac{c}{c_0} \quad ,$$

where c_0 is non-negative, is to be considered. We let c_0, c have the joint probability density $f(c_0, c)$ and we wish to find the distribution of r from this.

The Jacobian of the transformation

$$c = r\, c_0 \quad , \quad c_0 = c_0$$

is

$$\left| \frac{\partial(c_0, c)}{\partial(c_0, r)} \right| = c_0 \quad .$$

Thus the joint probability density for c_0 and r is

$$c_0 \, f(c_0, r\, c_0) \, .$$

The density for the distribution of r can be obtained as a marginal density by integrating out c_0 to give

$$(2.2.1) \qquad h(r) = \int_0^{\infty} c_0 \, f(c_0, r\, c_0) \, d\, c_0 \quad .$$

Let $M(T_0, T)$ be the joint moment generating function of c_0 and c. Then

$$M(T_0,T) = \mathcal{E}(e^{T_0 c_0 + Tc})$$

$$= \int_{-\infty}^{\infty} \int_0^{\infty} e^{T_0 c_0 + Tc} \, f(c_0,c) \, d\, c_0 \, d\, c \quad .$$

By the Fourier inversion formula we have

$$f(c_0,c) = \frac{1}{(2\pi i)^2} \iint M(T_0,T) e^{-(T_0 c_0 + Tc)} \, d\, T_0 \, d\, T \, .$$

The paths of integration are along the imaginary axes in the T_0 and T planes from $-i\infty$ to $i\infty$ or along any allowable deformation of these paths. That is, along any path, $\alpha - i\infty$ to $\alpha + i\infty$, such that no singularities of $M(T_0,T)$ lie on the new path of integration or between it and the imaginary axis. Since $c = r\, c_0$, we have

$$f(c_0, r\, c_0) = \frac{1}{(2\pi i)^2} \iint M(T_0,T) e^{-c_0(T_0 + r\, T)} \, d\, T_0 \, d\, T \, .$$

Considering the linear transformation

$$u = T_0 + r\,T \quad , \quad T = T$$

with Jacobian

$$\left| \frac{\partial(T_0, T)}{\partial(u,\, T)} \right| = 1 \quad ,$$

it is seen that this equation becomes

$$f(c_0, r\,c_0) = \frac{1}{(2\pi i)^2} \iint M(u-r\,T,\,T) e^{-c_0 u}\, du\, dT \;.$$

The integration of u is taken over a path in the u-plane corresponding to that of T_0 in the T_0-plane. Then

$$\int_0^\infty f(c_0, rc_0) e^{c_0 u}\, d\,c_0 = \int_0^\infty \left[\frac{1}{(2\pi i)^2} \iint M(u-rT,T) e^{-c_0 u} du\, dT \right] e^{c_0 u}\, d\,c_0$$

$$= \frac{1}{2\pi i} \int \left\{ \int_0^\infty \left[\frac{1}{2\pi i} \int M(u-rT,T) e^{-c_0 u}\, du \right] e^{c_0 u}\, dc_0 \right\} dT$$

Using the Fourier inversion formula we see that

$$\int_0^\infty \left[\frac{1}{2\pi i} \int M(u-rT,\, T) e^{-c_0 u}\, du \right] e^{c_0 u}\, d\,c_0 = M(u-rT,\, T)$$

so that

$$\int_0^\infty f(c_0,\, r\,c_0)\, e^{c_0 u}\, d\,c_0 = \frac{1}{2\pi i} \int M(u-rT,\, T)\, dT \;.$$

Differentiating, where permissible, under the integral sign with respect to u, we obtain

$$\int_0^\infty f(c_0,\ r\ c_0)\ c_0\ e^{c_0 u}\ d\ c_0 = \frac{1}{2\pi i} \int \frac{\partial}{\partial u} [M(u-rT,\ T)]\ dT\ .$$

Putting $u = 0$ in this equation, we have, by equation (2.2.1),

$$h(r) = \frac{1}{2\pi i} \int \frac{\partial}{\partial u} [M(u-rT,\ T)] \Big|_{u=0} dT\ .$$

This is equation (2.5) of Daniels' paper.

If we wish to transform T to some other variable z by $T = T(z,u)$ then

$$M(u-rT,\ T)\ dT = M[u-rT(z,u),\ \ T(z,u)] \frac{\partial T(z,u)}{\partial z}\ dz\ .$$

For simplicity we write

$$M[u-rT(z,u),\ T(z,u)] \frac{\partial T}{\partial z} (z,u)\ dz = M(u-rT,T) \frac{\partial T}{\partial z}\ dz\ .$$

Then we have

$$\int_0^\infty f(c_0,r\ c_0)\ e^{c_0 u}\ d\ c_0 = \frac{1}{2\pi i} \int M(u-rT,T) \frac{\partial T}{\partial z}\ dz\ .$$

Integration is along the transformed contour in the z-plane. By differentiating with respect to u under the sign of integration and setting $u = 0$, we obtain

$$(2.2.2) \qquad h(r) = \frac{1}{2\pi i} \int \frac{\partial}{\partial u} [M(u-rT,T) \frac{\partial T}{\partial z}\ ] \Big|_{u=0} dz\ .$$

This equation, known as the Cramer-Geary inversion formula, corresponds to equation (2.7) of Daniels' paper and is the form used in the work to follow.

The main problem of the paper is to evaluate approximately integrals of the above type when the statistics c and c_0 are calculated from moderately large samples. The cases considered are found to have integrands which can be written in the form

$$\varphi(z)\ [\psi(z)]^n$$

where n is the sample size.

In the applications considered, $\varphi(z)$ may be expanded about $z = \hat{z}$, where

$$\psi'(\hat{z}) = 0 \ ,$$

and the resulting series integrated term by term. It is found that this leads to an asymptotic expansion in powers of (n^{-1}) the dominant term of which is used as the approximation.

3. Non-circular Markov Process: Known Mean

As noted by Daniels [(2), page 175], "In testing for independence the use of a circularly defined sample serial correlation coefficient may be objectionable on grounds of loss of power and sensitivity to extraneous trends. But when $\rho \neq 0$ the circular

definition of the process itself is artificial and can only be justified if the results arrived at by its use are not substantially affected by the assumption of circularity." Hence, we shall direct our attention to the sample serial correlation coefficient with no circularity assumptions.

Consider the linear Markov process

$$x_s = \rho x_{s-1} + e_s \quad , \quad s = 1, 2, \ldots, n \quad ,$$

where $e_1, \ldots, e_n$ are independent, $N(0,1)$ random variables. It has been shown that the process may be considered stationary if $|\rho| < 1$, where ρ is the serial correlation coefficient. Assuming this to be true we have

$$\mathrm{Cov}(x_s, x_{s-\tau}) = \frac{\rho^{|\tau|}}{1-\rho^2} \quad .$$

Let $x_1, \ldots, x_n$ be a sample of observations from the stationary process. They have a joint multivariate normal distribution since $e_1, \ldots, e_n$ are independent, $N(0,1)$ random variables. The variance-covariance matrix of $x_1, \ldots, x_n$ is the $(n \times n)$ matrix

$$\underline{C} = \begin{bmatrix} \frac{1}{1-\rho^2} & \frac{\rho}{1-\rho^2} & \cdots & \frac{\rho^{n-1}}{1-\rho^2} \\ \frac{\rho}{1-\rho^2} & \frac{1}{1-\rho^2} & \cdots & \frac{\rho^{n-2}}{1-\rho^2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\rho^{n-1}}{1-\rho^2} & \frac{\rho^{n-2}}{1-\rho^2} & \cdots & \frac{1}{1-\rho^2} \end{bmatrix} .$$

To obtain the value of the determinant $|\underline{C}|$ and the inverse of $\underline{C}$, $\underline{C}^{-1}$, we proceed as follows:

$$|\underline{C}| = \begin{vmatrix} \frac{1}{1-\rho^2} & \frac{\rho}{1-\rho^2} & \cdots & \frac{\rho^{n-1}}{1-\rho^2} \\ \frac{\rho}{1-\rho^2} & \frac{1}{1-\rho^2} & \cdots & \frac{\rho^{n-2}}{1-\rho^2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\rho^{n-1}}{1-\rho^2} & \frac{\rho^{n-2}}{1-\rho^2} & \cdots & \frac{1}{1-\rho^2} \end{vmatrix}_n$$

By multiplying the second column by ρ and subtracting it from the first, we obtain

$$|\underline{C}| = \begin{vmatrix} 1 & \frac{\rho}{1-\rho^2} & \cdots & \frac{\rho^{n-1}}{1-\rho^2} \\ 0 & \frac{1}{1-\rho^2} & \cdots & \frac{\rho^{n-2}}{1-\rho^2} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \frac{\rho^{n-2}}{1-\rho^2} & \cdots & \frac{1}{1-\rho^2} \end{vmatrix}_n$$

$$= \begin{vmatrix} \frac{1}{1-\rho^2} & \frac{\rho}{1-\rho^2} & \cdots & \frac{\rho^{n-2}}{1-\rho^2} \\ \frac{\rho}{1-\rho^2} & \frac{1}{1-\rho^2} & \cdots & \frac{\rho^{n-3}}{1-\rho^2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\rho^{n-2}}{1-\rho^2} & \frac{\rho^{n-3}}{1-\rho^2} & \cdots & \frac{1}{1-\rho^2} \end{vmatrix}_{n-1} .$$

Proceeding in the same manner, the value of $|\underline{C}|$ is found to be

$$(2.3.1)\quad |\underline{C}| = \begin{vmatrix} \frac{1}{1-\rho^2} & \frac{\rho}{1-\rho^2} \\ \frac{\rho}{1-\rho^2} & \frac{1}{1-\rho^2} \end{vmatrix}_2 = \frac{1}{1-\rho^2} \quad .$$

Now,

$$\underline{C}^{-1} = (c_{jk}) = \left(\frac{\text{adj } C_{kj}}{|\underline{C}|} \right) \quad .$$

Adj.C_{kj} is obtained in the same manner that we obtained $|\underline{C}|$. Since $\underline{C}$ is symmetric,

$$c_{jk} = c_{kj} \qquad (j \neq k) \quad .$$

Thus, we find $\underline{C}^{-1}$ to be the $(n \times n)$ matrix

$$(2.3.2)\quad \underline{C}^{-1} = \begin{bmatrix} 1 & -\rho & & & & \\ -\rho & 1+\rho^2 & -\rho & & (0) & \\ & -\rho & 1+\rho^2 & -\rho & & \\ & & \cdot & \cdot & \cdot & \\ (0) & & & \cdot & \cdot & \cdot \\ & & & & 1+\rho^2 & -\rho \\ & & & & -\rho & 1 \end{bmatrix} .$$

The joint distribution of $x_1, \ldots, x_n$ is given by

$$d\,F(x_1,\ldots,x_n) = \frac{1}{(2\pi)^{\frac{n}{2}}} \frac{1}{|\underline{C}|^{\frac{1}{2}}} \exp\{-\tfrac{1}{2}\,\underline{x}'\,\underline{C}^{-1}\,\underline{x}\}\,dx_1 \ldots dx_n \quad ,$$

where $\underline{C}$ is the variance-covariance matrix and

$$\underline{x}' = (x_1, \ldots, x_n) \quad .$$

For the case under consideration, where $|\underline{C}|$ and $\underline{C}^{-1}$ are given by equations (2.3.1) and (2.3.2) respectively, we have

$$d\,F(x_1,\ldots,x_n) = \frac{(1-\rho^2)^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} \exp\{-\tfrac{1}{2}[x_1^2 + (1+\rho^2)(x_2^2 + \ldots + x_{n-1}^2)$$

$$+ x_n^2 - 2\rho(x_1 x_2 + \ldots + x_{n-1} x_n)]\}\,dx_1 \ldots dx_n \ .$$

The sample estimate of ρ is taken to be

$$r = \frac{c}{c_0} \quad ,$$

where

$$c = x_1 x_2 + \ldots + x_{n-1} x_n \quad ,$$

$$c_0 = \tfrac{1}{2}\,x_1^2 + x_2^2 + \ldots + x_{n-1}^2 + \tfrac{1}{2}\,x_n^2 \quad .$$

The joint moment generating function of c and c_0 is

$$M(T_0,T) = \mathcal{E}\left(e^{T_0 c_0 + Tc}\right)$$

$$= \frac{(1-\rho^2)^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} e^{T_0 c_0 + Tc - \frac{1}{2}\,\underline{x}'\,\underline{C}^{-1}\,\underline{x}}\,dx_1 \ldots dx_n .$$

Now,

$$T_0c_0 + Tc - \tfrac{1}{2}\,\underline{x}'\,\underline{C}^{-1}\,\underline{x} = -\tfrac{1}{2}[(1-T_0)x_1^2 + (1+\rho^2-2T_0)(x_2^2 + \dots + x_{n-1}^2)$$

$$+ (1-T_0)x_n^2 - 2(\rho+T)(x_1x_2 + \dots + x_{n-1}x_n)]$$

$$= -\tfrac{1}{2}\,\underline{x}'\,\underline{B}\,\underline{x} \quad ,$$

where $\underline{B}$ is the $(n \times n)$ matrix

$$(2.3.4) \qquad \underline{B} = \begin{bmatrix} 1-T_0 & -(\rho+T) & & & & \\ -(\rho+T) & 1+\rho^2-2T_0 & -(\rho+T) & & (0) & \\ & -(\rho+T) & 1+\rho^2-2T_0 & -(\rho+T) & & \\ & & \cdot & \cdot & \cdot & \\ & (0) & & \cdot & \cdot & \cdot \\ & & & & 1+\rho^2-2T_0 & -(\rho+T) \\ & & & & -(\rho+T) & 1-T_0 \end{bmatrix} .$$

Then $M(T_0,T)$ may be written as

$$M(T_0,T) = \frac{(1-\rho^2)^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} e^{-\frac{1}{2}\,\underline{x}'\,\underline{B}\,\underline{x}}\, dx_1 \dots dx_n \,.$$

There is an orthogonal transformation that will diagonalize $\underline{B}$.

Making this transformation and integration we get

$$(2.3.5) \qquad M(T_0,T) = \frac{(1-\rho^2)^{\frac{1}{2}}}{|\underline{B}|^{\frac{1}{2}}}$$

where $|\underline{B}|^{-\frac{1}{2}}$ is the Jacobian of the transformation.

$|\underline{B}|$ is evaluated as follows: Letting

$$\begin{aligned} 1 - T_0 &= a \\ 1 + \rho^2 - 2T_0 &= b \\ -(\rho + T) &= c \ , \end{aligned}$$

we have

$$|\underline{B}| = \begin{vmatrix} a & c & & & & & \\ c & b & c & & & (0) & \\ & c & b & c & & & \\ & & \cdot & \cdot & \cdot & & \\ & (0) & & \cdot & \cdot & \cdot & \\ & & & & & b & c \\ & & & & & c & a \end{vmatrix}_n$$

$$= \begin{vmatrix} b+(a-b) & c & & & & & \\ c & b & c & & & (0) & \\ & c & b & c & & & \\ & & \cdot & \cdot & \cdot & & \\ & (0) & & \cdot & \cdot & \cdot & \\ & & & & & b & c \\ & & & & & c & a \end{vmatrix}_n$$

$$= \begin{vmatrix} b & c & & & & & \\ c & b & c & & & (0) & \\ & c & b & c & & & \\ & & \cdot & \cdot & \cdot & & \\ & (0) & & \cdot & \cdot & \cdot & \\ & & & & & b & c \\ & & & & & c & a \end{vmatrix}_n + (a-b) \begin{vmatrix} b & c & & & & \\ c & b & c & & (0) & \\ & & \cdot & \cdot & \cdot & \\ (0) & & & \cdot & \cdot & \cdot \\ & & & & b & c \\ & & & & c & a \end{vmatrix}_{n-1} .$$

Similarly, by replacing a by $b+(a-b)$ in the lower right hand corner, we obtain

$$|\underline{B}| = G_n + 2(a-b)\, G_{n-1} + (a-b)^2\, G_{n-2} \;,$$

where

$$G_n = \begin{vmatrix} b & c & & & & & \\ c & b & c & & & (0) & \\ & c & b & c & & & \\ & & \cdot & \cdot & \cdot & & \\ & (0) & & \cdot & \cdot & \cdot & \\ & & & & & b & c \\ & & & & & c & b \end{vmatrix}_n$$

Since

$$a-b = -(\rho^2 - T_0) \;,$$

the value of $|\underline{B}|$ is given by

$$(2.3.6) \qquad |\underline{B}| = G_n - 2(\rho^2 - T_0)\, G_{n-1} + (\rho^2 - T_0)^2\, G_{n-2} \;.$$

It can be seen that

$$G_n = b \begin{vmatrix} b & c & & & & \\ c & b & c & & (0) & \\ & \cdot & \cdot & \cdot & & \\ (0) & & \cdot & \cdot & \cdot & \\ & & & & b & c \\ & & & & c & b \end{vmatrix}_{n-1} - c \begin{vmatrix} c & & & & (0) & \\ c & b & c & & & \\ & \cdot & \cdot & \cdot & & \\ (0) & & \cdot & \cdot & \cdot & \\ & & & & b & c \\ & & & & c & b \end{vmatrix}_{n-1}$$

$$= b\, G_{n-1} - c^2\, G_{n-2} \;.$$

That is,

$$(2.3.7) \quad G_{n+2} - b\, G_{n+1} + c^2\, G_n = 0$$

Letting E be the forward difference operator of the calculus of finite differences defined by

$$(2.3.8) \quad E[f(j)] = f(j+1) ,$$

equation (2.3.7) may be written as

$$(2.3.9) \quad (E^2 - b\,E + c^2)\, G_n = 0 .$$

The auxiliary equation of equation (2.3.9),

$$x^2 - bx + c^2 = 0 ,$$

has the roots

$$u = \frac{b + \sqrt{b^2 - 4c^2}}{2}$$

and

$$v = \frac{b - \sqrt{b^2 - 4c^2}}{2} ,$$

where

$$u + v = b$$

and

$$u\,v = c^2 .$$

Hence, the general solution of (2.3.9) is

$$G_n = A_1\, u^n + A_2\, v^n .$$

It is now necessary to evaluate A_1 and A_2. Evaluating G_n for $n = 0$ and $n = 1$ we have

$$G_0 = 1$$

and

$$G_1 = b$$

respectively, so that

$$G_0 = A_1 + A_2 = 1$$

and

$$G_1 = A_1 u + A_2 v = b .$$

From this it is found that

$$A_1 = \frac{u}{u-v}$$

and

$$A_2 = \frac{-v}{u-v}$$

and hence, the solution of equation (2.3.9) is

$$(2.3.10) \quad G_n = \frac{u^{n+1} - v^{n+1}}{u-v} .$$

Let

$$z + \frac{1}{z} = \frac{1 + \rho^2 - 2 T_0}{\rho + T}$$
$$= - \left(\frac{b}{c} \right)$$

so that

$$(2.3.11) \quad z^2 + \frac{b}{c} z + 1 = 0 .$$

The roots of equation (2.3.11) are

$$z = \frac{-b + \sqrt{b^2 - 4c^2}}{2c} = -\frac{v}{c}$$

and

$$\frac{1}{z} = \frac{-b - \sqrt{b^2 - 4c^2}}{2c} = -\frac{u}{c} .$$

Hence,

$$v = -c\,z ,$$

$$u = -\frac{c}{z}$$

and

$$u-v = -\frac{c}{z}(1 - z^2) .$$

Substituting these values of u and v in equation (2.3.10), the value of G_n in terms of z is

$$(2.3.12) \quad G_n = \frac{(-c)^n}{z^n(1-z^2)} \{1 - z^{2n+2}\} .$$

By equations (2.3.6) and (2.3.12) and replacing a, b and c by their values, we have

$$|\underline{B}| = \frac{(\rho+T)^n}{z^n(1-z^2)}[1-z^{2n+2}] - \frac{2(\rho^2-T_0)(\rho+T)^{n-1}}{z^{n-1}(1-z^2)}[1-z^{2n}]$$

$$+ \frac{(\rho^2-T_0)^2(\rho+T)^{n-2}}{z^{n-2}(1-z^2)}[1 - z^{2n-2}] .$$

On collecting terms this reduces to

$$(2.3.13)\quad |\underline{B}| = \frac{(\rho+T)^n}{z^n(1-z^2)} \{[1 - \frac{z(\rho^2-T_0)}{\rho+T}]^2 - z^{2n}[z - \frac{(\rho^2-T_0)}{\rho+T}]^2\} .$$

As mentioned in Daniels' paper, it can be shown that, if $(1-r^2)$ is not small, leaving out the term in z^{2n} within the bracket in equation (2.3.13) incurs an exponentially small error in the final approximation. Omitting the term in z^{2n}, the approximate value of $|\underline{B}|$ is

$$(2.3.14)\quad |\underline{B}| \sim \frac{(\rho+T)^n}{z^n(1-z^2)} [1 - \frac{z(\rho^2-T_0)}{\rho+T}]^2 .$$

Now, let

$$(2.3.15)\quad u = T_0 + r\,T$$

so that

$$z + \frac{1}{z} = \frac{1+\rho^2-2T_0}{\rho+T} = \frac{1+\rho^2-2u+2rT}{\rho+T}$$

or

$$(2.3.16)\quad z + \frac{1}{z} = \frac{1 + \rho^2 - 2\rho r - 2u}{\rho+T} + 2r .$$

Hence,

$$\frac{1 - 2rz + z^2}{z} = \frac{1 - 2\rho r + \rho^2 - 2u}{\rho+T}$$

and

$$\text{(2.3.17)} \quad \rho+T = \frac{z(1 - 2\rho r + \rho^2 - 2u)}{1 - 2rz + z^2} .$$

From equation (2.3.17) we see that T may be written as

$$\text{(2.3.18)} \quad T = - \left[\frac{(1-\rho z)(\rho-z) + 2uz}{1-2rz + z^2}\right] .$$

By use of equations (2.3.15), (2.3.17) and (2.3.18), it is seen that

$$1 - \frac{z(\rho^2-T_0)}{\rho+T} = 1 - \frac{(1-2rz+z^2)(\rho^2-u+rT)}{1-2\rho r+\rho^2-2u}$$

$$= \frac{(1-2\rho r+\rho^2-2u)-(\rho^2-u)(1-2rz+z^2)+r[(1-\rho z)(\rho-z)+2uz]}{1-2\rho r+\rho^2-2u}$$

or

$$\text{(2.3.19)} \quad 1 - \frac{z(\rho^2-T_0)}{\rho+T} = \frac{(1-\rho z)[1-\rho r-(r-\rho)z]-u(1-z^2)}{1-2\rho r+\rho^2-2u} .$$

Using equations (2.3.17) and (2.3.19) in equation (2.3.14), the approximate value of $|\underline{B}|$ is found to be

$$|\underline{B}| \sim \frac{z^n(1-2\rho r+\rho^2-2u)^n}{z^n(1-z^2)(1-2rz+z^2)^n} \left\{\frac{(1-\rho z)[1-\rho r-(r-\rho)z]-u(1-z^2)}{1-2\rho r+\rho^2-2u}\right\}^2$$

$$= \frac{(1-2\rho r+\rho^2-2u)^{n-2}}{(1-z^2)(1-2rz+z^2)^n} \{(1-\rho z)[1-\rho r-(r-\rho)z]-u(1-z^2)\}^2 .$$

Substituting this result in equation (2.3.5) gives

$$(2.3.20) \quad M(u-rT,T) \sim \frac{(1-\rho^2)^{\frac{1}{2}}(1-z^2)^{\frac{1}{2}}(1-2rz+z^2)^{\frac{n}{2}}}{(1-2\rho r+\rho^2-2u)^{\frac{n}{2}-1}\{(1-\rho z)[1-\rho r-(r-\rho)z]-u(1-z^2)\}}$$

with $\quad z + \frac{1}{z} = \frac{1+\rho^2-2T_0}{\rho+T}$.

Although some errors were present in equation (6.4) of Daniels' paper, equation (2.3.20) above and equation (6.5) of Daniels' paper are the same.

Since equation (2.3.17) holds, we have

$$(2.3.21) \quad \frac{\partial T}{\partial z} = \frac{(1-z^2)(1-2\rho r+\rho^2-2u)}{(1-2rz+z^2)^2} .$$

Combining equations (2.3.20) and (2.3.21), we obtain

$$M(u-rT,T)\frac{\partial T}{\partial z} \sim \frac{(1-z^2)^{\frac{3}{2}}(1-\rho^2)^{\frac{1}{2}}(1-2rz+z^2)^{\frac{n}{2}-2}}{(1-2\rho r+\rho^2-2u)^{\frac{n}{2}-2}\{(1-\rho z)[1-\rho r-(r-\rho)z]-u(1-z^2)\}}$$

and hence,

$$\frac{\partial}{\partial u}[M(u-rT,T)\frac{\partial T}{\partial z}] \sim \frac{(1-z^2)^{\frac{3}{2}}(1-\rho^2)^{\frac{1}{2}}(1-2rz+z^2)^{\frac{n}{2}-2}}{(1-2\rho r+\rho^2-2u)^{\frac{n}{2}-1}\{(1-\rho z)[1-\rho r-(r-\rho)z]-u(1-z^2)\}}$$

$$\times \{n-4 + \frac{(1-z^2)(1-2\rho r+\rho^2-2u)}{\{(1-\rho z)[1-\rho r-(r-\rho)z]-u(1-z^2)\}}\}$$

so that

$$\frac{\partial}{\partial u}[M(u-rT,T)\frac{\partial T}{\partial z}]\Bigg|_{u=0} \sim \frac{n(1-z^2)^{\frac{3}{2}}(1-\rho^2)^{\frac{1}{2}}(1-2rz+z^2)^{\frac{n}{2}-2}}{(1-2\rho r+\rho^2)^{\frac{n}{2}-1}\{(1-\rho z)[1-\rho r-(r-\rho)z]\}}$$

$$\times \{ 1 + \frac{1}{n} [\frac{(1-z^2)(1-2\rho r+\rho^2)}{(1-\rho z)[1-\rho r-(r-\rho)z]} - 4]\} .$$

By equation (2.2.2), we have

$$(2.3.22)\quad h(r) \sim \frac{n(1-\rho^2)^{\frac{1}{2}}}{2\pi i(1-2\rho r+\rho^2)^{\frac{n}{2}-1}} \int \varphi(z)\,(1-2rz+z^2)^{\frac{n}{2}-2}\,dz \;,$$

where

$$(2.3.23)\quad \varphi(z) = \frac{(1-z^2)^{\frac{3}{2}}}{(1-\rho z)[1-\rho r-(r-\rho)z]} \left\{1 + \frac{1}{n}\left[\frac{(1-z^2)(1-2\rho r+\rho^2)}{(1-\rho z)[1-\rho r-(r-\rho)z]} - 4\right]\right\} .$$

With $u = 0$, the mapping

$$z + \frac{1}{z} = \frac{1+\rho^2+2rT}{\rho+T}$$

is easily seen to consist of the following elementary mappings:

$$A:\quad V = \rho + T$$

$$B:\quad U = \frac{V}{1-2\rho r+\rho^2}$$

$$C:\quad S = \frac{1}{2U}$$

$$D:\quad Q = S + r$$

$$E:\quad z + \frac{1}{z} = 2Q \;.$$

A maps the T-plane cut along the parts of the real axis exterior to the interval $\left\{\frac{-(1+\rho)^2}{2(1+r)}, \frac{(1-\rho)^2}{2(1-r)}\right\}$ onto the V-plane cut along the parts of the real axis exterior to the interval $\left\{\frac{-(1-2\rho r+\rho^2)}{2(1+r)}, \frac{(1-2\rho r+\rho^2)}{2(1-r)}\right\}$.

B maps the region of the V-plane, as above, onto the U-plane cut along the real axis exterior to the interval $\left\{\frac{-1}{2(1+r)}, \frac{1}{2(1-r)}\right\}$.

C maps this region of the U-plane onto the S- plane cut along the real axis from $-(1+r)$ to $(1-r)$. This region of the S-plane is mapped onto the Q-plane cut along the real axis from -1 to 1 by D, and this region of the Q-plane is mapped onto the interior of the unit circle, $|z| = 1$, in the z-plane by E. Hence, the net effect of the transformation is to map the whole T-plane cut along the real axis exterior to the interval $\{ \frac{-(1+\rho)^2}{2(1+r)} , \frac{(1-\rho)^2}{2(1-r)} \}$ onto the interior of the unit circle.

To obtain the end-points of the path of integration in the z-plane, we note that as T goes to $\pm i \infty$, $z + \frac{1}{z}$ goes to $2r$ since, by equation (2.3.16),

$$z + \frac{1}{z} \Bigg|_{u=0} = \frac{1 + \rho^2 - 2\rho r}{\rho + T} + 2r \quad .$$

If

$$z + \frac{1}{z} = 2r$$

then

$$z^2 - 2rz + 1 = 0$$

and

$$z = r \pm i \sqrt{1 - r^2}$$

or

$$z = e^{\pm i\theta} \quad ,$$

where

$$r = \cos\theta.$$

We also see that if $T = 0$

$$z + \frac{1}{z} = \rho + \frac{1}{\rho}$$

and hence, the transformed path in the z-plane cuts the real axis at $z = \rho$. Thus, the path of integration in the z-plane crosses the real axis at $z = \rho$ and ends on the boundary of the unit circle at $e^{-i\theta}$ and $e^{i\theta}$, where $r = \cos\theta$.

The only possible singularity in the integrand of the integral for $h(r)$ would be the real singularity arising from

$$1 - \rho r - (r-\rho)z = 0 .$$

This singularity is avoided if, on the real axis, we confine z to the following range:

$$\text{(i)} \quad \rho \leq z \leq r , \quad \rho < r$$

Now, we may write

$$1-\rho r-(r-\rho)z = (1-z)(r-\rho) + (1-r)(1+\rho) .$$

Then, since

$$|\rho| < 1 \quad \text{and} \quad |r| < 1 ,$$

we see that

$$1-z > 0, \quad r-\rho > 0, \quad 1-r > 0 \quad \text{and} \quad 1+\rho > 0 .$$

From this it can be seen that

$$(1-z)(r-\rho) > 0 \quad \text{and} \quad (1-r)(1+\rho) > 0 \ ,$$

and finally, that

$$1 - \rho r - (r-\rho)z > 0 \quad .$$

(ii) $r \leq z \leq \rho \ , \ r < \rho$

Here,

$$1-\rho r-(r-\rho)z = (1-r^2) + (\rho-r)(z-r) \ .$$

As

$$|\rho| < 1 \quad \text{and} \quad |r| < 1 \ ,$$

we have

$$1-r^2 > 0, \quad \rho-r > 0 \quad \text{and} \quad z-r \geq 0 \quad .$$

Then

$$(\rho-r)(z-r) \geq 0$$

and hence

$$1 - \rho r - (r-\rho)z > 0 \quad .$$

(iii) $z = r = \rho$

Since

$$|\rho| < 1 \quad \text{and} \quad |r| < 1 \quad ,$$

$$1 - \rho r > 0 \quad \text{and} \quad (r-\rho)z = 0$$

so that

$$1 - \rho r - (r-\rho)z > 0 \quad .$$

Since there is no singularity in the closed interval joining ρ and r , we may deform the path of integration in the z-plane to be the straight line joining $e^{-i\theta}$ to $e^{i\theta}$ and crossing the real axis at $z = r$.

On the path of integration we may write

$$(2.3.24) \quad z = r + i\,w(1-r^2)^{\frac{1}{2}} \quad ,$$

where w is the real variable with

$$-1 \leq w \leq 1 \quad .$$

Then

$$1 - 2rz + z^2 = (1-r^2)(1-w^2)$$

and

$$dz = i(1-r^2)^{\frac{1}{2}}\,dw \quad .$$

Substituting equation (2.3.24) in equation (2.3.22), we have

$$(2.3.25) \quad h(r) \sim \frac{n(1-\rho^2)^{\frac{1}{2}}(1-r^2)^{\frac{n-3}{2}}}{2\pi\,(1-2\rho r+\rho^2)^{\frac{n}{2}-1}} \int_{-1}^{1} \varphi(z)(1-w^2)^{\frac{n}{2}-2}\,dw \quad .$$

The integrand of $h(r)$ is of the form

$$\varphi(z)\,[\psi(z)]^{\frac{n}{2}-2}$$

where

$$\psi(z) = 1 - 2rz + z^2 \quad .$$

Since the integral cannot be readily evaluated in closed form, we expand $\varphi(z)$ as a power series in $(z-r)$, where

$$\psi'(r) = 0 \; ,$$

and integrate with respect to w. Using equation (2.3.24), we have

$$z - r = i\, w(1-r^2)^{\frac{1}{2}}$$

and hence,

$$\varphi(z) = \varphi(r) + \varphi'(r)\,(z-r) + \ldots + \frac{\varphi^{(k)}(r)\,(z-r)^k}{k!} + \ldots$$

$$= \varphi(r) + iw(1-r^2)^{\frac{1}{2}}\varphi'(r) + \ldots + \frac{i^k\, w^k\,(1-r^2)^{\frac{k}{2}}}{k!}\,\varphi^{(k)}(r) + \ldots \quad .$$

Substituting this in equation (2.3.25), we have

$$\text{(2.3.26)} \quad h(r) \sim \frac{n(1-\rho^2)^{\frac{1}{2}}(1-r^2)^{\frac{n-3}{2}}}{2\pi(1-2\rho r+\rho^2)^{\frac{n}{2}-1}} \sum_{k=0}^{\infty} \frac{i^k(1-r^2)^{\frac{k}{2}}}{k!}\,\varphi^{(k)}(r) \int_{-1}^{1} w^k(1-w^2)^{\frac{n}{2}-2}\,dw \; .$$

Now, if

$$k = 2m+1 \; , \quad m = 0,\, 1,\, 2,\, \ldots \quad ,$$

then

$$w^k(1-w^2)^{\frac{n}{2}-2}$$

is an odd function and the integral vanishes. Therefore, equation (2.3.26) reduces to

$$(2.3.27)\quad h(r) \sim \frac{n(1-\rho^2)^{\frac{1}{2}}(1-r^2)^{\frac{n-3}{2}}}{2\pi(1-2\rho r+\rho^2)^{\frac{n}{2}-1}} \sum_{k=0}^{\infty} \frac{(-1)^k(1-r^2)^k}{(2k)!}\varphi^{(2k)}(r)\int_{-1}^{1} w^{2k}(1-w^2)^{\frac{n}{2}-2}\,dw\ .$$

To evaluate

$$(2.3.28)\quad I_k = \int_{-1}^{1} w^{2k}\,(1-w^2)^{\frac{n}{2}-2}\,dw$$

we proceed in the following manner: Letting

$$w^2 = u\ ,\quad dw = \frac{1}{2}u^{-\frac{1}{2}}\,du\ ,$$

we have

$$I_k = \int_0^1 u^{k-\frac{1}{2}}\,(1-u)^{\frac{n}{2}-2}\,du = \beta(k+\tfrac{1}{2}\,,\,\tfrac{n}{2}-1)$$

$$= \frac{\Gamma(k+\frac{1}{2})\,\Gamma(\frac{n}{2}-1)}{\Gamma(k+\frac{n}{2}-\frac{1}{2})}\ .$$

Thus, we have

$$\text{for}\quad k=0\ ,\quad I_0 = \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n-1}{2})} = \frac{\sqrt{\pi}\,\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n-1}{2})}\ ;$$

$$k=1\ ,\quad I_1 = \frac{\Gamma(\frac{3}{2})\,\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n+1}{2})} = \frac{1}{n-1}\,I_0\ ;$$

$$k=2\ ,\quad I_2 = \frac{\Gamma(\frac{5}{2})\,\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n+3}{2})} = \frac{1.3}{(n-1)(n+1)}\,I_0\ ;$$

and in general, for $k = 1, 2, \ldots$

$$(2.3.29)\quad I_k = \frac{1.3\ldots(2k-1)}{(n-1)(n+1)\ldots(n+2k-3)}\, I_0 \;,$$

where

$$I_0 = \frac{\sqrt{\pi}\,\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n-1}{2})} \quad .$$

Substituting equation (2.3.29) in equation (2.3.27), we obtain

$$h(r) \sim \frac{n(1-\rho^2)^{\frac{1}{2}}(1-r^2)^{\frac{n-3}{2}}\, I_0}{2\pi(1-2\rho r+\rho^2)^{\frac{n}{2}-1}} \{\varphi(r) + \sum_{k=1}^{\infty} \frac{(-1)^k(1-r^2)^k}{(2k)!}\, \varphi^{(2k)}(r) \frac{[1.3\ldots(2k-1)]}{[(n-1)\ldots(n+2k-3)]}\}$$

or

$$(2.3.30)\quad h(r) \sim \frac{n}{2\sqrt{\pi}}\, \frac{\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n-1}{2})}\, \frac{(1-\rho^2)^{\frac{1}{2}}(1-r^2)^{\frac{n-3}{2}}}{(1-2\rho r+\rho^2)^{\frac{n}{2}-1}} \{\varphi(r)+\sum_{k=1}^{\infty} \frac{(-1)^k(1-r^2)^k\;\;\varphi^{(2k)}(r)}{2^k k!\,[(n-1)\ldots(n+2k-3)]}\}$$

since

$$(2k)! = 2^k\, k!\, [(2k-1)\ldots 3.1] \;.$$

By replacing z by r in equation (2.3.23), we have

$$\varphi(r) = \frac{(1-r^2)^{\frac{1}{2}}}{(1-\rho r)} \{1 + \frac{1}{n}[\frac{1-2\rho r+\rho^2}{1-\rho r} - 4]\}$$

$$= \frac{(1-r^2)^{\frac{1}{2}}}{(1-\rho r)} \{1 + O(n^{-1})\} \quad .$$

Then a first approximation to $h(r)$ is

$$(2.3.31)\quad h(r) \sim \frac{n}{2\sqrt{\pi}} \frac{\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n-1}{2})} \frac{(1-\rho^2)^{\frac{1}{2}}(1-r^2)^{\frac{n}{2}-1}}{(1-\rho r)(1-2\rho r+\rho^2)^{\frac{n}{2}-1}} \{1 + O(n^{-1})\} \quad .$$

This corresponds to equation (6.7) of Daniels' paper.

4. Accuracy of the Approximation and Renormalization of $h(r)$

In the expression for $h(r)$, given by equation (2.3.31), the remainder is relatively $O(n^{-1})$ though an exponentially small term has also been neglected. We assume n to be large enough for this term to be negligible. Some insight into the errors involved is obtained by a heuristic discussion of orders of magnitude.

In what follows we show that the variance of r is $O(n^{-1})$ and values of $h(r)$ outside some range on either side of ρ, which is $O(n^{-\frac{1}{2}})$, are negligibly small. Then, over the effective range of r, $(r-\rho)$ can be taken as $O(n^{-\frac{1}{2}})$.

From equation (2.3.31) we see that

$$h(\rho) \sim \frac{n}{2\sqrt{\pi}} \frac{\Gamma(\frac{n}{2} - 1)}{\Gamma(\frac{n - 1}{2})} \frac{1}{(1-\rho^2)^{\frac{1}{2}}} \{1 + O(n^{-1})\} \quad .$$

Now, by Stirling's formula,

$$\Gamma(x) = e^{-x} x^{x - \frac{1}{2}} (2\pi)^{\frac{1}{2}} \{1 + O(x^{-1})\} \quad ,$$

we have

$$\frac{\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n-1}{2})} = e^{\frac{1}{2}} (1- \frac{1}{n-1})^{\frac{n-1}{2}} (1- \frac{1}{n-1})^{-\frac{1}{2}} (\frac{n-2}{2})^{-\frac{1}{2}} \{1 + O(n^{-1})\} \quad .$$

As n increases,

$$(1 - \frac{1}{n-1})^{-\frac{1}{2}} \sim 1 ,$$

$$(1 - \frac{1}{n-1})^{\frac{n-1}{2}} \sim e^{-\frac{1}{2}} ,$$

$$(\frac{n-2}{2})^{-\frac{1}{2}} \sim (\frac{2}{n})^{\frac{1}{2}}$$

and

$$h(\rho) \sim [\frac{n}{2\pi\ (1-\rho^2)}]^{\frac{1}{2}} \{1 + O(n^{-1})\} .$$

Hence, $h(r)$ is $O(n^{\frac{1}{2}})$ at $r = \rho$.

To establish a bound on the order of magnitude of $\mathcal{E}[(r-\rho)^2]$, consider

$$\frac{(1-r^2)^{\frac{n}{2}-1}}{(1-\rho r)(1-2\rho r+\rho^2)^{\frac{n}{2}-1}} = \frac{1}{(1-\rho r)} [\frac{1-r^2}{1-2\rho r+\rho^2}]^{\frac{n}{2}-1}$$

$$= \frac{1}{(1-\rho r)} [1 - \frac{(r-\rho)^2}{1-2\rho r+\rho^2}]^{\frac{n}{2}-1}$$

$$\leq \frac{1}{(1-|\rho|)} [1 -(\frac{r-\rho}{1+|\rho|})^2]^{\frac{n}{2}-1} .$$

Also, as shown above,

$$\frac{n}{2\sqrt{\pi}} \frac{\Gamma(\frac{n}{2} - 1)}{\Gamma(\frac{n-1}{2})}$$

is $O(n^{\frac{1}{2}})$.

Let

$$g(r) = \frac{n}{2\sqrt{\pi}} \frac{\Gamma(\frac{n}{2} - 1)}{\Gamma(\frac{n-1}{2})} \frac{(1-\rho^2)^{\frac{1}{2}} (1-r^2)^{\frac{n}{2}-1}}{(1-\rho r)(1-2\rho r+\rho^2)^{\frac{n}{2}-1}}$$

so that

$$h(r) \sim g(r) \, [1 + O(n^{-1})] \quad .$$

Then it follows from the foregoing discussion that

$$g(r) \le O(n^{\frac{1}{2}}) \, [1 - (\frac{r-\rho}{1+|\rho|})^2]^{\frac{n}{2}-1}$$

and

$$0 < \mathcal{E}[(r-\rho)^2] = \int_{-1}^{1} (r-\rho)^2 \, h(r) \, dr$$

$$\sim \int_{-1}^{1} (r-\rho)^2 \, g(r) \, dr$$

$$\le O(n^{\frac{1}{2}}) \int_{-1}^{1} (r-\rho)^2 \, [1 - (\frac{r-\rho}{1+|\rho|})^2]^{\frac{n}{2}-1} \, dr \quad .$$

Let

$$\frac{r-\rho}{1+|\rho|} = u \, ,$$

then

$$\int_{-1}^{1} (r-\rho)^2 g(r) dr \le O(n^{\frac{1}{2}}) \int_{\frac{-(1+\rho)}{1+|\rho|}}^{\frac{1-\rho}{1+|\rho|}} u^2 \, (1-u^2)^{\frac{n}{2}-1} \, du$$

$$\leq O(n^{\frac{1}{2}}) \int_0^1 u^2 \, (1-u^2)^{\frac{n}{2}-1} \, du$$

$$= O(n^{\frac{1}{2}}) \frac{\Gamma(\frac{3}{2}) \, \Gamma(\frac{n}{2})}{\Gamma(\frac{n+3}{2})}$$

$$= O(n^{-1}) \, .$$

This suggests that the distribution of r is concentrated about $r = \rho$ within a range which is $O(n^{-\frac{1}{2}})$ on either side of ρ.

From the above discussion, we have

$$\frac{(1-r^2)^{\frac{n}{2}-1}}{(1-\rho r)(1-2\rho r+\rho^2)^{\frac{n}{2}-1}} \leq \frac{1}{(1-|\rho|)} \exp\left\{(\tfrac{n}{2}-1) \log[1 - (\frac{r-\rho}{1+|\rho|})^2]\right\} \, .$$

It is known that if $|x| < 1$

$$\log(1-x) = - x - \frac{1}{2} x^2 - \frac{1}{3} x^3 - \ldots \, .$$

Then, since

$$(\frac{r-\rho}{1+|\rho|})^2 < 1 \, ,$$

$$\exp\{(\tfrac{n}{2} - 1)\log[1 - (\frac{r-\rho}{1+|\rho|})^2]\} = \exp\{(\tfrac{n}{2} - 1)[- (\frac{r-\rho}{1+|\rho|})^2 - \frac{1}{2}(\frac{r-\rho}{1+|\rho|})^4 - \ldots]$$

$$\leq \exp[- (\tfrac{n}{2} - 1)(\frac{r-\rho}{1+|\rho|})^2] \, .$$

Hence,

$$\frac{(1-r^2)^{\frac{n}{2}-1}}{(1-\rho r)(1-2\rho r+\rho^2)^{\frac{n}{2}-1}} < \frac{1}{(1-|\rho|)} \exp[-(\tfrac{n}{2}-1)(\frac{r-\rho}{1+|\rho|})^2] \quad .$$

Thus, we have

$$g(r) < O(n^{\frac{1}{2}}) \exp[-(\tfrac{n}{2}-1)(\frac{r-\rho}{1+|\rho|})^2] \quad .$$

Suppose that $|r-\rho|$ is $O(n^{\frac{\alpha}{2}})$. Then

$$(\tfrac{n}{2}-1)(\frac{r-\rho}{1+|\rho|})^2$$

is $O(n^{\alpha+1})$ and

$$\exp[-(\tfrac{n}{2}-1)(\frac{r-\rho}{1+|\rho|})^2]$$

will tend to zero exponentially as $n \to \infty$ provided $\alpha > -1$. Thus $g(r)$, and hence $h(r)$, is exponentially small for values of r outside some range on either side of ρ which is $O(n^{-\frac{1}{2}})$. We conclude that over the effective range of r, $(r-\rho)$ may be considered $O(n^{-\frac{1}{2}})$.

We now show that if r is replaced by ρ in the first neglected term in the expansion of $h(r)$ the term is altered by an amount $O(n^{-\frac{3}{2}})$. Considering equation (2.3.30), we see that

$$h(r) \sim \frac{n}{2\sqrt{\pi}} \frac{\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n-1}{2})} \frac{(1-\rho^2)^{\frac{1}{2}}(1-r^2)^{\frac{n-3}{2}}}{(1-2\rho r+\rho^2)^{\frac{n}{2}-1}} \varphi(r)[1 - \frac{(1-r^2)\varphi''(r)}{2(n-1)\varphi(r)} + O(n^{-2})] \quad ,$$

where

$$\varphi(r) = \frac{(1-r^2)^{\frac{1}{2}}}{(1-\rho r)} \{1 + \frac{1}{n}[\frac{1-2\rho r+\rho^2}{1-\rho r} - 4]\} \quad .$$

We may write

$$\varphi(r) = \frac{(1-r^2)^{\frac{1}{2}}}{(1-\rho r)} \{1 + \frac{1}{n} [\frac{1 - r^2 + (r-\rho)^2}{1-\rho r} - 4]\}$$

$$= \frac{(1-r^2)^{\frac{1}{2}}}{(1-\rho r)} \{1 + \frac{1}{n} [\frac{1-r^2}{1-\rho r} - 4] + O(n^{-2})\}$$

since $(r-\rho)^2$ is $O(n^{-1})$. Then

$$(2.4.1) \quad h(r) \sim \frac{n}{2\sqrt{\pi}} \frac{\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n-1}{2})} \frac{(1-r^2)^{\frac{n-3}{2}}}{(1-2\rho r+\rho^2)^{\frac{n}{2}-1}} \frac{(1-\rho^2)^{\frac{1}{2}} (1-r^2)^{\frac{1}{2}}}{(1-\rho r)}$$

$$x \{1 + \frac{1}{n} [\frac{1-r^2}{1-\rho r} - 4] - \frac{(1-r^2)}{2(n-1)} \frac{\varphi''(r)}{\varphi(r)} + O(n^{-2})\} .$$

On expanding

$$\frac{(1-\rho^2)^{\frac{1}{2}} (1-r^2)^{\frac{1}{2}}}{1-\rho r}$$

in a Taylor's series about $r = \rho$, we find that

$$\frac{(1-\rho^2)^{\frac{1}{2}} (1-r^2)^{\frac{1}{2}}}{1-\rho r} = 1 - \frac{(r-\rho)^2}{2(1-\rho^2)^2} + O[(r - \rho)^3] ,$$

and on expanding

$$\frac{1-r^2}{1-2\rho r+\rho^2}$$

in a Taylor's series about the same point, we have

$$\frac{1-r^2}{1-2\rho r+\rho^2} = 1 - \frac{(r-\rho)^2}{1-\rho^2} + O[(r-\rho)^3] .$$

The binomial expansion of

$$\left(\frac{1-r^2}{1-2\rho r+\rho^2}\right)^{\frac{1}{2(1-\rho^2)}} = \{1 - \frac{(r-\rho)^2}{1-\rho^2} + O[(r-\rho)^3]\}^{\frac{1}{2(1-\rho^2)}}$$

yields

$$\left(\frac{1-r^2}{1-2\rho r+\rho^2}\right)^{\frac{1}{2(1-\rho^2)}} = 1 - \frac{(r-\rho)^2}{2(1-\rho^2)^2} + O[(r-\rho)^3]$$

so that

$$\frac{(1-\rho^2)^{\frac{1}{2}}(1-r^2)^{\frac{1}{2}}}{1-\rho r} = \left(\frac{1-r^2}{1-2\rho r+\rho^2}\right)^{\frac{1}{2(1-\rho^2)}} + O[(r-\rho)^3]$$

and

$$(2.4.2) \qquad \frac{(1-\rho^2)^{\frac{1}{2}}(1-r^2)^{\frac{1}{2}}}{1-\rho r} = \left(\frac{1-r^2}{1-2\rho r+\rho^2}\right)^{\frac{1}{2(1-\rho^2)}} [1 + O(n^{-\frac{3}{2}})]$$

since $(r-\rho)$ is $O(n^{-\frac{1}{2}})$. Substituting the result (2.4.2) in equation (2.4.1), we obtain

$$h(r) \sim \frac{n}{2\sqrt{\pi}} \frac{\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n-1}{2})} \frac{(1-r^2)^{\frac{N-1}{2}}}{(1-2\rho r+\rho^2)^{\frac{N}{2}}}$$

$$\times \{ 1 + \frac{1}{n} [\frac{1-r^2}{1-\rho r} - 4] - \frac{(1-r^2)\ \varphi''(r)}{2(n-1)\ \varphi(r)} + O(n^{-\frac{3}{2}}) \} ,$$

where

$$N = n-1 + \frac{\rho^2}{1-\rho^2} .$$

Let

$$q(r) = 1 + \frac{1}{n} [\frac{1-r^2}{1-\rho r} - 4] - \frac{(1-r^2)\varphi''(r)}{2(n-1)\varphi(r)} .$$

Expanding $q(r)$ in a Taylor's series about $r = \rho$ gives

$$q(r) = q(\rho) + q'(\rho)(r-\rho) + \ldots \quad .$$

Now,

$$q(\rho) = 1 - \frac{3}{n} - \frac{(1-\rho^2)\varphi''(\rho)}{2(n-1)\varphi(\rho)}$$

is $O(1)$ and $q'(\rho)$ is $O(n^{-1})$. Thus, since $(r-\rho)$ is $O(n^{-\frac{1}{2}})$,

$$q(r) = q(\rho) + O(n^{-\frac{3}{2}})$$

$$= q(\rho)\,[1 + O(n^{-\frac{3}{2}})] \; .$$

But, since $q(\rho)$ is independent of r, it becomes part of the normalizing constant. Hence,

$$h(r) \sim K \frac{(1-r^2)^{\frac{N-1}{2}}}{(1-2\rho r+\rho^2)^{\frac{N}{2}}} \; [1 + O(n^{-\frac{3}{2}})] \; ,$$

where K is an adjusted normalizing constant.

Employing the above results, we shall now proceed to renormalize $h(r)$. First, consider

$$J_k = \int_{-1}^{1} \frac{(1-r^2)^{\frac{k-1}{2}}}{(1-2\rho r+\rho^2)^{\frac{k}{2}}} \, dr \quad , \quad k = 1, 2, \ldots \quad ,$$

and let

$$u = \frac{1 + r}{2} \quad .$$

Then

$$J_k = \frac{2^k}{(1+\rho)^k} \int_0^1 u^{\frac{k-1}{2}} (1-u)^{\frac{k-1}{2}} \left[1 - \frac{4\rho u}{(1+\rho)^2}\right]^{-\frac{k}{2}} du \quad .$$

It can be shown by [(15), page 293, example no. 1] that

$$(2.4.3) \qquad J_k = \frac{2^k}{(1+\rho)^k} \frac{[\Gamma(\frac{k+1}{2})]^2}{\Gamma(k+1)} F\left[\frac{k}{2}, \frac{k+1}{2}; k+1; \frac{4\rho}{(1+\rho)^2}\right] ,$$

where $F[a,b; c; z]$ is the usual hypergeometric function. It follows from [(4), page 64, formula 24] and the hypergeometric power series that

$$F\left[\frac{k}{2}, \frac{k+1}{2}; k+1; \frac{4\rho}{(1+\rho)^2}\right] = (1+\rho)^k F\left[\frac{k}{2}, 0; \frac{k}{2}+1; \rho^2\right]$$

$$= (1+\rho)^k .$$

Then equation (2.4.3) becomes

$$J_k = \frac{2^k [\Gamma(\frac{k+1}{2})]^2}{\Gamma(k+1)} \quad .$$

Using the duplication formula for the gamma function, [(15), page 240], we have

$$J_k = \frac{2^k [\Gamma(\frac{k+1}{2})]^2}{2^k (\pi)^{-\frac{1}{2}} \Gamma(\frac{k+1}{2}) \Gamma(\frac{k}{2}+1)}$$

$$= \frac{\sqrt{\pi}\, \Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2}+1)} \quad .$$

Thus we have shown for $k = 1, 2, \ldots$

$$(2.4.4) \qquad J_k = \int_{-1}^{1} \frac{(1-r^2)^{\frac{k-1}{2}}}{(1-2\rho r+\rho^2)^{\frac{k}{2}}} dr = \frac{\sqrt{\pi}\, \Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2}+1)} \quad .$$

Employing this general result, we see that

$$I = \int_{-1}^{1} \frac{(1-r^2)^{\frac{N-1}{2}}}{(1-2\rho r+\rho^2)^{\frac{N}{2}}} \, dr = \frac{\sqrt{\pi}\,\Gamma(\frac{N+1}{2})}{\Gamma(\frac{N}{2}+1)} .$$

Then the renormalized density function is given by

$$h(r) \sim K(n,\rho) \frac{(1-r^2)^{\frac{N-1}{2}}}{(1-2\rho r+\rho^2)^{\frac{N}{2}}} [1 + O(n^{-\frac{3}{2}})] ,$$

where

$$K(n,\rho) = \frac{1}{I} = \frac{\Gamma(\frac{N}{2}+1)}{\sqrt{\pi}\,\Gamma(\frac{N+1}{2})}$$

and

$$N = n-1 + \frac{\rho^2}{1-\rho^2} .$$

This is Leipnik's form of $h(r)$ and corresponds to equation (7.4) of Daniels' paper.

5. Non-circular Markov Process. Unknown Mean

We consider briefly the case in which the mean is unknown and give Daniels' results. We do not use this exact method in our work which follows in Chapter III.

When the mean is unknown, it is estimated by

$$\bar{x} = \frac{\frac{1}{2} x_1 + x_2 + \ldots + x_{n-1} + \frac{1}{2} x_n}{n-1} .$$

The serial correlation coefficient, ρ , is estimated by

$$r = \frac{C}{C_0} ,$$

where

$$C = (x_1-\bar{x})(x_2-\bar{x}) + \ldots + (x_{n-1}-\bar{x})(x_n-\bar{x}) = c - (n-1)\,\bar{x}^2 ,$$

$$C_0 = \tfrac{1}{2}(x_1-\bar{x})^2+(x_2-\bar{x})^2 + \ldots + (x_{n-1}-\bar{x})^2+\tfrac{1}{2}(x_n-\bar{x})^2=c_0-(n-1)\bar{x}^2 ,$$

with c , c_0 defined as in section 3 of this chapter.

The joint moment generating function of C and C_0 is

$$M(T_0,T) = \mathcal{E}\,(e^{T_0C_0+TC})$$

$$= \frac{(1-\rho^2)^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{T_0C_0+TC-\frac{1}{2}\underline{x}'\underline{C}^{-1}\underline{x}}\, dx_1 \ \ldots \ dx_n ,$$

where $\underline{C}^{-1}$ is given by equation (2.3.2). Since

$$T_0C_0+TC-\tfrac{1}{2}\ \underline{x}'\ \underline{C}^{-1}\ \underline{x} = T_0c_0+Tc-\tfrac{1}{2}\ \underline{x}'\ \underline{C}^{-1}\ \underline{x} - (n-1)\ (T_0+T)\ \bar{x}^2$$

$$= -\tfrac{1}{2}\ \underline{x}'\ \underline{B}\ \underline{x} - \frac{(T_0+T)}{n-1}\ \underline{x}'\ \underline{m}\ \underline{m}'\ \underline{x}$$

$$= -\tfrac{1}{2}\ \underline{x}'\ [\underline{B} + \frac{2(T_0+T)}{n-1}\ \underline{m}\ \underline{m}']\ \underline{x} ,$$

with $\underline{B}$ given by (2.3.4),

$$\underline{x}' = (x_1, \ldots, x_n)$$

and

$$\underline{m}' = (\tfrac{1}{2}, 1, \ldots, 1, \tfrac{1}{2}) ,$$

we have

$$M(T_0,T) = \frac{(1-\rho^2)^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}\underline{x}'[\underline{B} + \frac{2(T_0+T)}{n-1}\underline{m}\,\underline{m}']\underline{x}} dx_1, \ldots, dx_n .$$

Then we have

$$M(T_0,T) = (1-\rho^2)^{\frac{1}{2}} \left|\underline{B} + \frac{2(T_0+T)}{n-1}\underline{m}\,\underline{m}'\right|^{-\frac{1}{2}} ,$$

which is obtained in a manner similar to equation (2.3.5).

On evaluating $\left|\underline{B} + \frac{2(T_0+T)}{n-1}\underline{m}\,\underline{m}'\right|$ and making the necessary substitutions and reductions, we obtain the following approximation after omitting a term in z^{n-1}:

$$M(u-rT,T) \sim \frac{(1-\rho^2)^{\frac{1}{2}}(1-z^2)^{\frac{1}{2}}(1-z)(1-2rz + z^2)^{\frac{n-1}{2}}}{(1-\rho)\{(1-\rho z)[1-\rho r-(r-\rho)z]-u(1-z^2)\}(1-2\rho r+\rho^2-2u)^{\frac{n-3}{2}}}$$

$$\times \left\{1 + \frac{(1+\rho)^2(1-2rz+z^2)[(z-\rho)(1-\rho z)(1-r)+u(1-z^2)]}{(n-1)(1-z^2)(1-2\rho r+\rho^2-2u)[1-\rho r-(r-\rho)z-u(1-z^2)]}\right\}$$

with

$$z + \frac{1}{z} = \frac{1 + \rho^2 - 2T_0}{\rho+T} .$$

To obtain an approximation with remainder relatively $O(n^{-\frac{3}{2}})$, we ignore the last factor and we take the dominant term as before in the expansion of the integral for $h(r)$. On renormalizing, we have

$$h(r) \sim \frac{K(1-\rho^2)^{\frac{1}{2}}(1-r^2)^{\frac{n-3}{2}}(1-r)}{(1-\rho r)(1-2\rho r + \rho^2)^{\frac{n-3}{2}}} \left[1 + O(n^{-\frac{3}{2}})\right] ,$$

where K is the normalizing constant. In Leipnik's form,

$$h(r) \sim \frac{2}{\sqrt{\pi}} \frac{\Gamma(\frac{N+3}{2})}{\Gamma(\frac{N}{2})[N(1-\rho)-(1+\rho)]} \frac{(1-r)(1-r^2)^{\frac{N}{2}-1}}{(1-2\rho r+\rho^2)^{\frac{N-1}{2}}} [1 + O(n^{-\frac{3}{2}})] ,$$

with

$$N = n - 1 + \frac{\rho^2}{1-\rho^2} .$$

CHAPTER III

DISTRIBUTION OF AN ESTIMATE OF THE SERIAL CORRELATION COEFFICIENT

1. Non-circular Markov Process. Known Mean

We consider a non-circular Markov process similar to that studied by Daniels. As given in [(2), (6.3)], Daniels takes the sample estimate of ρ to be

$$r = \frac{c}{c_0} ,$$

where

$$c = x_1 x_2 + \dots + x_{n-1} x_n ,$$

$$c_0 = \tfrac{1}{2} x_1^2 + x_2^2 + \dots + x_{n-1}^2 + \tfrac{1}{2} x_n^2 ,$$

whereas, we shall take the sample estimate of ρ to be

$$r = \frac{c}{c_0} ,$$

where

$$(3.1.1)\left\{\begin{array}{l} c = \tfrac{1}{2} x_1^2 + x_1 x_2 + \dots + x_{n-1} x_n + \tfrac{1}{2} x_n^2 , \\ c_0 = x_1^2 + \dots + x_n^2 . \end{array}\right.$$

The method employed in obtaining the approximate distribution of r is similar to that used by Daniels. In addition to discussing the case of the known mean, we shall discuss that of the unknown mean in the following section.

The joint distribution of $x_1, \dots, x_n$ is given by equation

(2.3.3) and the sample estimate of ρ is

$$r = \frac{c}{c_0} ,$$

where c and c_0 are given by (3.1.1). Then the joint moment generating function of c and c_0 is

$$M(T_0,T) = \mathcal{E}\left(e^{T_0c_0+Tc}\right)$$

$$= \frac{(1-\rho^2)^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{T_0c_0+Tc-\frac{1}{2}\,\underline{x}'\,\underline{C}^{-1}\,\underline{x}}\, dx_1 \;\ldots\; dx_n ,$$

where $\underline{C}^{-1}$ is given by result (2.3.2). Now,

$$T_0c_0+Tc-\tfrac{1}{2}\,\underline{x}'\,\underline{C}^{-1}\underline{x} = -\tfrac{1}{2}\,[(1-2T_0-T)x_1{}^2+(1+\rho^2-2T_0)(x_2{}^2+\ldots+x_{n-1}^2)$$

$$+(1-2T_0-T)x_n{}^2-2(\rho+T)(x_1\,x_2+\ldots+x_{n-1}\,x_n)]$$

$$= -\tfrac{1}{2}\,\underline{x}'\,\underline{D}\,\underline{x} ,$$

where

$$(3.1.2)\quad \underline{D} = \begin{bmatrix} 1-2T_0-T & -(\rho+T) & & & & & \\ -(\rho+T) & 1+\rho^2-2T_0 & -(\rho+T) & & & (0) & \\ & -(\rho+T) & 1+\rho^2-2T_0 & -(\rho+T) & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & (0) & & & & 1+\rho^2-2T_0 & -(\rho+T) \\ & & & & & -(\rho+T) & 1-2T_0-T \end{bmatrix}_n$$

and

$$\underline{x}' = (x_1, \ldots, x_n) .$$

Then

$$M(T_0,T) = \frac{(1-\rho^2)^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\frac{1}{2}\,\underline{x}'\,\underline{D}\,\underline{x}}\,dx_1 \;\cdots\; dx_n \,.$$

As mentioned in Chapter II, section 3, there is an orthogonal transformation of $x_1, \ldots, x_n$ that will diagonalize $\underline{D}$. Diagonalizing $\underline{D}$ and carrying out the integration, we obtain

$$(3.1.3) \qquad M(T_0,T) = (1-\rho^2)^{\frac{1}{2}}\,|\underline{D}|^{-\frac{1}{2}} \quad ,$$

where $|\underline{D}|^{-\frac{1}{2}}$ is the Jacobian of the transformation.

To evaluate $|\underline{D}|$ we proceed as follows: $(\rho+T)$ is factored from each row of $|\underline{D}|$ and the following substitutions are made:

$$a = \frac{1 - 2T_0 - T}{\rho+T} \quad ,$$

$$b = \frac{1 + \rho^2 - 2T_0}{\rho+T} \quad ,$$

so that $|\underline{D}|$ may be written as

$$(3.1.4) \qquad |\underline{D}| = (\rho+T)^n \; R_n \quad ,$$

where R_n is the determinant

$$R_n = \begin{vmatrix} a & -1 & & & & & \\ -1 & b & -1 & & & (0) & \\ & -1 & b & -1 & & & \\ & & \cdot & \cdot & \cdot & & \\ & (0) & & \cdot & \cdot & \cdot & \\ & & & & & b & -1 \\ & & & & & -1 & a \end{vmatrix}_n .$$

Let R^*_j $(j < n)$ be the leading principal minor of order j of the determinant R_n. Then

$$(3.1.5)\begin{cases} R^*_1 = a \ , \\ R^*_2 = ab-1 \end{cases}$$

and in general,

$$R^*_j = b\, R^*_{j-1} - R^*_{j-2}$$

or, equivalently,

$$(3.1.6) \quad R^*_{j+2} - b\, R^*_{j+1} + R^*_j = 0 \ .$$

Taking E to be the forward difference operator of the calculus of finite differences defined by equation (2.3.8) , we may write equation (3.1.6) as

$$(3.1.7) \quad (E^2 - bE + 1)\, R^*_j = 0 \ .$$

Solving this difference equation, we find that the roots of the auxiliary equation

$$z^2 - bz + 1 = 0$$

are

$$z = \frac{b + \sqrt{b^2 - 4}}{2}$$

and

$$\frac{1}{z} = \frac{b - \sqrt{b^2 - 4}}{2} ,$$

where

$$(3.1.8) \qquad z + \frac{1}{z} = b .$$

Hence the general solution of equation (3.1.7) is

$$(3.1.9) \qquad R^*_j = K_1 z^j + K_2 z^{-j} .$$

Applying conditions (3.1.5) to equation (3.1.9), we have

$$R^*_1 = a = K_1 z + K_2 z^{-1} ,$$

$$R^*_2 = ab-1 = K_1 z^2 + K_2 z^{-2} .$$

From these equations and equation (3.1.8), the values of K_1 and K_2 are found to be

$$K_1 = \frac{1-az}{1-z^2} , \quad K_2 = \frac{z(a-z)}{1-z^2} .$$

Substituting these values in equation (3.1.9), we have

$$R^*_j = \left(\frac{1-az}{1-z^2}\right) z^j + \left[\frac{z(a-z)}{1-z^2}\right] z^{-j} .$$

But,

$$R_n = a\,R^*_{n-1} - R^*_{n-2}$$

$$= a\left\{\left(\frac{1-az}{1-z^2}\right) z^{n-1} + \left[\frac{z(a-z)}{1-z^2}\right] z^{-(n-1)}\right\}$$

$$- \left\{\left(\frac{1-az}{1-z^2}\right) z^{n-2} + \left[\frac{z(a-z)}{1-z^2}\right] z^{-(n-2)}\right\}$$

or

$$(3.1.10) \quad R_n = \frac{1}{z^n(1-z^2)} \left[z^2(a-z)^2 - z^{2n-2}(1-az)^2\right] .$$

Hence, by equations (3.1.4) and (3.1.10), we have

$$(3.1.11) \quad |\underline{D}| = \frac{(\rho+T)^n}{z^n(1-z^2)} \left[z^2(a-z)^2 - z^{2n-2}(1-az)^2\right] ,$$

where

$$a = \frac{1 - 2T_0 - T}{\rho+T} ,$$

$$b = \frac{1 + \rho^2 - 2T_0}{\rho+T} = z + \frac{1}{z} .$$

Substituting equations (2.3.15), (2.3.17) and (2.3.18) for u, $(\rho+T)$ and T respectively, we see that

$$a = b - \left(\frac{\rho^2+T}{\rho+T}\right) = z + z^{-1} - \left(\frac{\rho^2+T}{\rho+T}\right) ,$$

$$z(a-z) = 1 - z\left(\frac{\rho^2+T}{\rho+T}\right)$$

$$= \frac{(1-\rho z)(1-2\rho r+\rho z+\rho-z)-2u(1-z)}{1-2\rho r+\rho^2-2u}$$

and

$$1 - az = z\left(\frac{\rho^2+T}{\rho+T}\right) - z^2$$

$$= \frac{(\rho-z)(\rho-2\rho rz-z+\rho z-1)-2uz(1-z)}{1-2\rho r+\rho^2-2u} .$$

Then equation (3.1.4) becomes

$$|\underline{D}| = \frac{(1-2\rho r+\rho^2-2u)^n}{(1-z^2)(1-2rz+z^2)^n} \{[\frac{(1-\rho z)(1-2\rho r+\rho z+\rho-z)-2u(1-z)}{1-2\rho r+\rho^2-2u}]^2$$

$$- z^{2n-2} [\frac{(\rho-z)(\rho-2\rho rz-z+\rho z-1)-2uz(1-z)}{1-2\rho r+\rho^2-2u}]^2 \} .$$

We omit the term in z^{2n-2} within the bracket which gives rise to an error that is exponentially small. Hence, on ignoring the term in z^{2n-2}, we have

$$|\underline{D}| \sim \frac{(1-2\rho r+\rho^2-2u)^{n-2}}{(1-z^2)(1-2rz+z^2)^n} [(1-\rho z)(1-2\rho r+\rho z+\rho-z)-2u(1-z)]^2 .$$

Thus, an approximation of $M(T_0,T)$, as given by equation (3.1.3), is

$$(3.1.12) \quad M(u-rT,T) \sim \frac{(1-\rho^2)^{\frac{1}{2}}(1-z^2)^{\frac{1}{2}}(1-2rz+z^2)^{\frac{n}{2}}}{(1-2\rho r+\rho^2-2u)^{\frac{n}{2}-1}[(1-\rho z)(1-2\rho r+\rho z+\rho-z)-2u(1-z)]} .$$

Combining equations (3.1.12) and (2.3.21) gives

$$M(u-rT,T)\frac{\partial T}{\partial z} \sim \frac{(1-\rho^2)^{\frac{1}{2}}(1-z^2)^{\frac{3}{2}}(1-2rz+z^2)^{\frac{n}{2}-2}}{(1-2\rho r+\rho^2-2u)^{\frac{n}{2}-2}[(1-\rho z)(1-2\rho r+\rho z+\rho-z)-2u(1-z)]}$$

and

$$\frac{\partial}{\partial u}\left[M(u-rT,T)\frac{\partial T}{\partial z}\right] \sim \frac{(1-\rho^2)^{\frac{1}{2}}(1-z^2)^{\frac{3}{2}}(1-2rz+z^2)^{\frac{n}{2}-2}}{(1-2\rho r+\rho^2-2u)^{\frac{n}{2}-1}\left[(1-\rho z)(1-2\rho r+\rho z+\rho-z)-2u(1-z)\right]}$$

$$\times \left\{ n - 4 + \frac{2(1-z)(1-2\rho r+\rho^2-2u)}{[(1-\rho z)(1-2\rho r+\rho z+\rho-z)-2u(1-z)]} \right\} .$$

Thus,

$$\frac{\partial}{\partial u}\left[M(u-rT,T)\frac{\partial T}{\partial z}\right]\Bigg|_{u=0} \sim \frac{(1-\rho^2)^{\frac{1}{2}}(1-z^2)^{\frac{3}{2}}(1-2rz+z^2)^{\frac{n}{2}-2}}{(1-2\rho r+\rho^2)^{\frac{n}{2}-1}(1-\rho z)(1-2\rho r+\rho z+\rho-z)}$$

$$\times \left[n - 4 + \frac{2(1-z)(1-2\rho r+\rho^2)}{(1-\rho z)(1-2\rho r+\rho z+\rho-z)}\right] .$$

By equation (2.2.2), we have

$$(3.1.13)\quad h(r) \sim \frac{n(1-\rho^2)^{\frac{1}{2}}}{2\pi i(1-2\rho r+\rho^2)^{\frac{n}{2}-1}} \int \varphi(z)\,(1-2rz+z^2)^{\frac{n}{2}-2}\,dz \quad ,$$

where

$$(3.1.14)\quad \varphi(z) = \frac{(1-z^2)^{\frac{3}{2}}}{(1-\rho z)(1-2\rho r+\rho z+\rho-z)}\left\{1 + \frac{1}{n}\left[\frac{2(1-z)(1-2\rho r+\rho^2)}{(1-\rho z)(1-2\rho r+\rho z+\rho-z)} - 4\right]\right\} .$$

As shown in Chapter II, section 3, the transformation

$$z + \frac{1}{z} = \frac{1+\rho^2+2rT}{\rho+T}$$

maps the T-plane cut exterior to the interval $\left\{ \frac{-(1+\rho)^2}{2(1+r)} , \frac{(1-\rho)^2}{2(1-r)} \right\}$ on the real axis onto the interior of the unit circle, $|z| = 1$, in the z-plane. Also, the path of integration in the z-plane crosses the real axis at $z = \rho$ and terminates on the boundary of the unit circle at $e^{-i\theta}$ and $e^{i\theta}$, where $r = \cos\theta$.

The real singularity arising from

$$1 - 2\rho r + \rho z + \rho - z = 0$$

is the only possible singularity in the integrand of result (3.1.13). On the real axis, we confine z to the following range so as to avoid this singularity:

(i) $\rho \leq z \leq r \;, \quad \rho < r$

We may write

$$1-2\rho r+\rho z+\rho-z = (1-\rho)(r-z) + (1+\rho)(1-r) \quad .$$

Since

$$|\rho| < 1 \text{ and } |r| < 1 \quad ,$$

then

$$1-\rho > 0, \quad r-z \geq 0, \quad 1+\rho > 0 \text{ and } 1-r > 0 \quad .$$

Hence,

$$(1-\rho)(r-z) \geq 0 \text{ and } (1+\rho)(1-r) > 0 \quad ,$$

and

$$1-2\rho r+\rho z+\rho-z > 0 \quad .$$

(ii) $r \leq z \leq \rho \;, \quad r < \rho$

Since

$$z \geq r, \quad |\rho| < 1 \text{ and } |r| < 1 \quad ,$$

$$\rho z \geq \rho r$$

and hence,

$$1-2\rho r+\rho z \geq 1-\rho r > 0 \; .$$

Also

$$\rho - z \geq 0 \; .$$

Then,

$$1-2\rho r+\rho z+\rho-z > 0 .$$

(iii) $z = r = \rho$

Here,

$$1-2\rho r+\rho z+\rho-z \;=\; 1-\rho^2 > 0 ,$$

since

$$|\rho| < 1 .$$

Because there is no singularity in the closed interval joining ρ and r, we may deform the path of integration in the z-plane to be the straight line joining $e^{-i\theta}$ to $e^{i\theta}$ and crossing the real axis at $z = r$.

As in Chapter II, section 3, we substitute equation (2.3.24) for z on the path of integration so that equation (3.1.13) becomes

$$h(r) \sim \frac{n(1-\rho^2)^{\frac{1}{2}}(1-r^2)^{\frac{n-3}{2}}}{2\pi\,(1-2\rho r+\rho^2)^{\frac{n}{2}-1}} \int_{-1}^{1} \varphi(z)\,(1-w^2)^{\frac{n}{2}-2}\,dw \;.$$

Following the method of Daniels (2) discussed in the previous chapter, we expand $\varphi(z)$ as a power series in $(z-r)$ and integrate with respect to w. By using equation (2.3.24), it is seen that

$$\varphi(z) = \varphi(r) + iw(1-r^2)^{\frac{1}{2}}\varphi'(r) + \ldots + \frac{i^k w^k (1-r^2)^{\frac{k}{2}}}{k!}\,\varphi^{(k)}(r) + \ldots$$

and hence,

$$(3.1.15)\quad h(r) \sim \frac{n}{2\pi}\,\frac{(1-\rho^2)^{\frac{1}{2}}(1-r^2)^{\frac{n-3}{2}}}{(1-2\rho r+\rho^2)^{\frac{n}{2}-1}} \sum_{k=0}^{\infty} \frac{i^k(1-r^2)^{\frac{k}{2}}}{k!}\,\varphi^{(k)}(r)\int_{-1}^{1} w^k(1-w^2)^{\frac{n}{2}-2}\,dw \;.$$

For the case that k is an odd integer, the integrand is an odd function, and the integral vanishes. Therefore, we may write equation (3.1.15) as

$$h(r) \sim \frac{n}{2\pi} \frac{(1-\rho^2)^{\frac{1}{2}} (1-r^2)^{\frac{n-3}{2}}}{(1 - 2\rho r + \rho^2)^{\frac{n}{2}-1}} \sum_{k=0}^{\infty} \frac{(-1)^k (1-r^2)^k}{(2k)!} \varphi^{(2k)}(r) \int_{-1}^{1} w^{2k}(1-w^2)^{\frac{n}{2}-2} dw.$$

Since

$$I_k = \int_{-1}^{1} w^{2k} (1-w^2)^{\frac{n}{2}-2} dw$$

is the same as equation (2.3.28), the value of I_k is given by (2.3.29). Substituting this value in the expression for $h(r)$, we have

$$h(r) \sim \frac{n}{2\pi} \frac{(1-\rho^2)^{\frac{1}{2}}(1-r^2)^{\frac{n-3}{2}}}{(1 - 2\rho r + \rho^2)^{\frac{n}{2}-1}} I_0\{\varphi(r) + \sum_{k=1}^{\infty} \frac{(-1)^k(1-r^2)^k}{(2k)!} \varphi^{(2k)}(r)\frac{[1.3...(2k-1)]}{[(n-1)...(n+2k-3)]}\}$$

or

$$h(r) \sim \frac{n}{2\sqrt{\pi}} \frac{(1-\rho^2)^{\frac{1}{2}}(1-r^2)^{\frac{n-3}{2}}}{(1 - 2\rho r + \rho^2)^{\frac{n}{2}-1}} \frac{\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n-1}{2})} \{\varphi(r) + \sum_{k=1}^{\infty} \frac{(-1)^k(1-r^2)^k \varphi^{(2k)}(r)}{2^k k! [(n-1)...(n+2k-3)]}\} .$$

Putting $z = r$ in equation (3.1.14) gives

$$\varphi(r) = \frac{(1-r^2)^{\frac{3}{2}}}{(1-\rho r)(1+\rho)(1-r)} \{1 + \frac{1}{n} [\frac{2(1-2\rho r+\rho^2)}{(1-\rho r)(1+\rho)} - 4]\}$$

or

$$(3.1.16) \quad \varphi(r) = \frac{(1-r^2)^{\frac{3}{2}}}{(1-\rho r)(1+\rho)(1-r)} [1 + O(n^{-1})] .$$

The dominant term in this asymptotic expansion is now taken as the

asymptotic value of $h(r)$. Thus, to $O(n^{-1})$ we have

$$h(r) \sim \frac{n}{2\sqrt{\pi}} \frac{(1-\rho^2)^{\frac{1}{2}}(1-r^2)^{\frac{n-3}{2}}}{(1 - 2\rho r + \rho^2)^{\frac{n}{2}-1}} \frac{\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n-1}{2})} \varphi(r) \; [1 + O(n^{-1})] \quad ,$$

where $\varphi(r)$ is given by (3.1.16). As we are neglecting terms which are relatively $O(n^{-1})$, we may write

$$h(r) \sim \frac{n}{2\sqrt{\pi}} \frac{\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n-1}{2})} \frac{(1-\rho^2)^{\frac{1}{2}}(1-r^2)^{\frac{n}{2}}}{(1-2\rho r+\rho^2)^{\frac{n}{2}-1}(1-\rho r)(1-r)(1+\rho)} \; [1 + O(n^{-1})] \; .$$

Using result (2.4.2) and following the discussion of section 4 of the previous chapter, it is seen that $h(r)$ becomes

$$h(r) \sim K \frac{(1-r^2)^{\frac{N+1}{2}}}{(1-r)(1-2\rho r+\rho^2)^{\frac{N}{2}}} \; [1 + O(n^{-\frac{3}{2}})] \quad ,$$

where K is an adjusted normalizing constant and

$$N = n - 1 + \frac{\rho^2}{1-\rho^2} \quad .$$

To renormalize $h(r)$ we consider

$$I = \int_{-1}^{1} \frac{(1-r^2)^{\frac{N+1}{2}}}{(1-r)(1-2\rho r+\rho^2)^{\frac{N}{2}}} dr$$

$$= \int_{-1}^{1} \frac{(1+r)\;(1-r^2)^{\frac{N-1}{2}}}{(1-2\rho r + \rho^2)^{\frac{N}{2}}} dr$$

$$= \int_{-1}^{1} \frac{(1 - r^2)^{\frac{N-1}{2}}}{(1-2\rho r+\rho^2)^{\frac{N}{2}}} dr + \int_{-1}^{1} \frac{r(1 - r^2)^{\frac{N-1}{2}}}{(1-2\rho r+\rho^2)^{\frac{N}{2}}} dr \; .$$

Using result (2.4.4) with $k = N$, we have

$$\int_{-1}^{1} \frac{(1-r^2)^{\frac{N-1}{2}}}{(1-2\rho r+\rho^2)^{\frac{N}{2}}} dr = \frac{\sqrt{\pi}\ \Gamma(\frac{N+1}{2})}{\Gamma(\frac{N}{2}+1)} .$$

Integrating by parts gives

$$J = \int_{-1}^{1} \frac{r(1-r^2)^{\frac{N-1}{2}}}{(1-2\rho r+\rho^2)^{\frac{N}{2}}} dr = \frac{N\rho}{N+1} \int_{-1}^{1} \frac{(1-r^2)^{\frac{N+1}{2}}}{(1-2\rho r+\rho^2)^{\frac{N}{2}+1}} dr .$$

Then, by result (2.4.4) with $k = N+2$,

$$J = \frac{N\rho}{(N+1)} \frac{\sqrt{\pi}\ \Gamma(\frac{N+3}{2})}{\Gamma(\frac{N}{2}+2)} .$$

Hence,

$$I = \frac{\sqrt{\pi}\ \Gamma(\frac{N+1}{2})}{\Gamma(\frac{N}{2}+1)} + \frac{N\rho}{(N+1)} \frac{\sqrt{\pi}\ \Gamma(\frac{N+3}{2})}{\Gamma(\frac{N}{2}+2)}$$

$$= \frac{\sqrt{\pi}\ \Gamma(\frac{N+1}{2})}{2\,\Gamma(\frac{N}{2}+2)} [N(1+\rho) + 2] .$$

Then the renormalized density function, in Leipnik's form, is

$$(3.1.17) \quad h(r) \sim K(n,\rho) \frac{(1-r^2)^{\frac{N+1}{2}}}{(1-r)(1-2\rho r+\rho^2)^{\frac{N}{2}}} [1 + O(n^{-\frac{3}{2}})] ,$$

where

$$K(n,\rho) = \frac{1}{I} = \frac{2\,\Gamma(\frac{N}{2}+2)}{\sqrt{\pi}\ \Gamma(\frac{N+1}{2})[N(1+\rho)+2]}$$

and

$$N = n - 1 + \frac{\rho^2}{1-\rho^2} .$$

2. Non-circular Markov Process: Unknown Mean

For the case when the mean is unknown, we estimate it by

$$\bar{x} = \frac{x_1 + \dots + x_n}{n} ,$$

and the serial correlation coefficient, ρ , is estimated by

$$(3.2.1) \qquad r = \frac{C}{C_0} ,$$

where

$$(3.2.2) \begin{cases} C = \frac{1}{2}(x_1-\bar{x})^2+(x_1-\bar{x})(x_2-\bar{x}) + \dots + (x_{n-1}-\bar{x})(x_n-\bar{x})+\frac{1}{2}(x_n-\bar{x})^2 \\ \quad = c - n\,\bar{x}^2 , \\ C_0 = (x_1-\bar{x})^2 + \dots + (x_n-\bar{x})^2 \\ \quad = c_0 - n\,\bar{x}^2 \end{cases}$$

with c, c_0 defined by (3.1.1).

Since equation (2.3.3) gives the joint distribution of $x_1, \dots, x_n$ and equation (3.2.1) the sample estimate of ρ , the joint moment generating function of C and C_0 is

$$M(T_0,T) = \mathcal{E}(e^{T_0C_0+TC})$$

$$= \frac{(1-\rho^2)^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} e^{T_0C_0+TC-\frac{1}{2}\underline{x}'\,\underline{C}^{-1}\,\underline{x}}\, dx_1 \dots dx_n ,$$

where $\underline{C}^{-1}$ is given by equation (2.3.2) and

$$\underline{x}' = (x_1, \ldots, x_n) .$$

Now,

$$T_0C_0+TC-\tfrac{1}{2}\underline{x}'\underline{C}^{-1}\underline{x} = T_0c_0+Tc-\tfrac{1}{2}\underline{x}'\underline{C}^{-1}\underline{x} - \frac{(T_0+T)}{n}(x_1 + \ldots + x_n)^2$$

$$= -\tfrac{1}{2}\,\underline{x}'\,\underline{D}\,\underline{x} - \frac{(T_0+T)}{n}\,\underline{x}'\,\underline{1}\,\underline{1}'\,\underline{x}$$

$$= -\tfrac{1}{2}\,\underline{x}'\,[\underline{D} + \frac{2(T_0+T)}{n}\,\underline{1}\,\underline{1}']\,\underline{x} \quad ,$$

where $\underline{D}$ is given by equation (3.1.2) and

$$\underline{1}' = (1, \ldots , 1) .$$

Then

$$M(T_0,T) = \frac{(1-\rho^2)^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}\underline{x}'[\underline{D} + \frac{2(T_0+T)}{n}\,\underline{1}\,\underline{1}']\,\underline{x}}\, dx_1 \ldots dx_n .$$

Using an orthogonal transformation of the variables $x_1, \ldots, x_n$, as mentioned in section 3 of the previous chapter, we diagonalize $[\underline{D} + \frac{2(T_0+T)}{n}\,\underline{1}\,\underline{1}']$. On diagonalizing this matrix and carrying out the integration, we obtain

$$(3.2.3) \quad M(T_0,T) = (1-\rho^2)^{\frac{1}{2}}\, |\underline{D} + \frac{2(T_0+T)}{n}\,\underline{1}\,\underline{1}'|^{-\frac{1}{2}} \quad ,$$

where $|\underline{D} + \frac{2(T_0+T)}{n}\,\underline{1}\,\underline{1}'|^{-\frac{1}{2}}$ is the Jacobian of the transformation.

To evaluate $\left|\underline{D} + \frac{2(T_0+T)}{n} \underline{1}\,\underline{1}'\right|$, we let

$$a = 1 + \rho^2 - 2T_0 + \frac{2}{n}(T_0+T) ,$$

$$b = \frac{2}{n}(T_0+T) - (\rho+T) ,$$

$$c = \frac{2}{n}(T_0+T)$$

and

$$f = 1 - 2T_0 - T + \frac{2}{n}(T_0+T) ,$$

so that

$$\left|\underline{D} + \frac{2(T_0+T)}{n} \underline{1}\,\underline{1}'\right| = \begin{vmatrix} f & b & & & & & \\ b & a & b & & & & \\ & b & a & b & (c) & & \\ & & \cdot & \cdot & \cdot & & \\ (c) & & & \cdot & \cdot & \cdot & \\ & & & & & a & b \\ & & & & & b & f \end{vmatrix}_n .$$

$\left|\underline{D} + \frac{2(T_0+T)}{n} \underline{1}\,\underline{1}'\right|$ is the same form as (I.1) of Appendix I the value of which is given by equation (I.15). In this case,

$$(3.2.4)\begin{cases} r = b - c = -(\rho+T) , \\ s = a - c = 1 + \rho^2 - 2T_0 \\ \text{and} \\ f-a = -(\rho^2 + T) . \end{cases}$$

Letting

$$z + \frac{1}{z} = \frac{1 + \rho^2 - 2T_0}{\rho+T} = -\left(\frac{s}{r}\right) ,$$

we see that

$$(3.2.5)\begin{cases} u = \dfrac{s + \sqrt{s^2 - 4r^2}}{2} = -rz = (\rho+r)z \quad , \\[2ex] v = \dfrac{s - \sqrt{s^2 - 4r^2}}{2} = \dfrac{-r}{z} = \dfrac{(\rho+T)}{z} \end{cases}$$

and

$$(3.2.6)\begin{cases} 2r+s = \dfrac{-r}{z}(1-z)^2 = \dfrac{(\rho+T)(1-z)^2}{z} \quad , \\[2ex] u-v = \dfrac{r}{z}(1-z^2) = \dfrac{-(\rho+T)(1-z^2)}{z} \quad . \end{cases}$$

Using results (3.2.4), (3.2.5) and (3.2.6) in equation (I.15), the value of $|\underline{D} + \frac{2(T_0+T)}{n}\ \underline{1}\ \underline{1}'|$ is found to be

$$|\underline{D} + \frac{2(T_0+T)}{n}\ \underline{1}\ \underline{1}'| =$$

$$\frac{-(\rho+T)^{n-4}z^{n+2}}{(1-z^2)(1-z)^4}\{[(\rho+T)z-(\rho^2+T)]^2[\frac{(\rho+T)^2(1-z)^4}{z^2} + \frac{2(T_0+T)}{n}(\frac{(\rho+T)(1-z^2)}{z}+\frac{n(\rho+T)(1-z)^2}{z})]$$

$$+ \frac{2(\rho+T)(T_0+T)(1-z)^2}{nz}[(\rho+T)^2z^2 - (\rho^2+T)^2]\}$$

$$+ \frac{(\rho+T)^{n-4}}{z^{n-4}(1-z^2)(1-z)^4}\{[\frac{(\rho+T)}{z}-(\rho^2+T)]^2[\frac{(\rho+T)^2(1-z)^4}{z^2}+\frac{2(T_0+T)}{n}(\frac{n(\rho+T)(1-z)^2}{z}-\frac{(\rho+T)(1-z^2)}{z})]$$

$$+ \frac{2(\rho+T)(T_0+T)(1-z)^2}{nz}[\frac{(\rho+T)^2}{z^2} - (\rho^2 + T)^2]\}$$

$$+ \frac{4(T_0+T)(\rho+T)^{n-3}}{n(1-z)^4}[z^2\rho^2(1-\rho)^2] \; .$$

Since $|z| < 1$, terms in z^n and z^0 are small relative to those in z^{-n}. Also, terms that are $O(n^{-1})$ are small for large values of n.

As an approximation of $\left|\underline{D} + \frac{2(T_0+T)}{n} \underline{1}\,\underline{1}'\right|$, we eliminate these terms and obtain

$$(3.2.7) \quad \left|\underline{D} + \frac{2(T_0+T)}{n} \underline{1}\,\underline{1}'\right| \sim$$

$$\frac{(\rho+T)^{n-3}}{z^n(1-z^2)(1-z)^2} \{[z(\rho^2+T)-(\rho+T)]^2[(\rho+T)(1+z^2)-2z(\rho-T_0)]\} .$$

Substituting equations (2.3.15) and (2.3.17) for u and $(\rho+T)$ respectively, we see that

$$(\rho+T)(1+z^2)-2z(\rho-T_0) = (\rho+T)(1-2rz+z^2)+2z[u-\rho(1-r)]$$

$$= z(1-\rho)^2$$

and

$$(T+\rho^2)z - (\rho+T) = -[(\rho+T)(1-z) + \rho z(1-\rho)]$$

$$= -z \left[\frac{(1-\rho z)(1-2\rho r+\rho z+\rho-z)-2u(1-z)}{1-2rz+z^2}\right] .$$

Then equation (3.2.7) may be written

$$\left|\underline{D} + \frac{2(T_0+T)}{n} \underline{1}\,\underline{1}'\right| \sim$$

$$\frac{(1-\rho)^2(1-2\rho r+\rho^2-2u)^{n-3}}{(1-z^2)(1-z)^2(1-2rz+z^2)^{n-1}} [1-\rho z)(1-2\rho r+\rho z+\rho-z)-2u(1-z)]^2 .$$

If we substitute the approximate value of

$|\underline{D} + \frac{2(T_0+T)}{n} \underline{1}\ \underline{1}'|$ in equation (3.2.3) then

$$M(u-rT,T) \sim \frac{(1-\rho^2)^{\frac{1}{2}}(1-z^2)^{\frac{1}{2}}(1-z)(1-2rz+z^2)^{\frac{n-1}{2}}}{(1-\rho)(1-2\rho r+\rho^2-2u)^{\frac{n-3}{2}}[(1-\rho z)(1-2\rho r+\rho z+\rho-z)-2u(1-z)]} .$$

Combining this result with equation (2.3.21) gives

$$M(u-rT,T)\frac{\partial T}{\partial z} \sim \frac{(1-\rho^2)^{\frac{1}{2}}(1-z^2)^{\frac{3}{2}}(1-z)(1-2rz+z^2)^{\frac{n-5}{2}}}{(1-\rho)(1-2\rho r+\rho^2-2u)^{\frac{n-5}{2}}[(1-\rho z)(1-2\rho r+\rho z+\rho-z)-2u(1-z)]}$$

and hence,

$$\frac{\partial}{\partial u}[M(u-rT,T)\frac{\partial T}{\partial z}] \sim \frac{(1-\rho^2)^{\frac{1}{2}}(1-z^2)^{\frac{3}{2}}(1-z)(1-2rz+z^2)^{\frac{n-5}{2}}}{(1-\rho)(1-2\rho r+\rho^2-2u)^{\frac{n-3}{2}}[(1-\rho z)(1-2\rho r+\rho z+\rho-z)-2u(1-z)]}$$

$$\times \{ n-5 + \frac{2(1-z)(1-2\rho r+\rho^2-2u)}{[(1-\rho z)(1-2\rho r+\rho z+\rho-z)-2u(1-z)]} \} .$$

Thus,

$$\frac{\partial}{\partial u}[M(u-rT,T)\frac{\partial T}{\partial z}]\Big|_{u=0} \sim \frac{(1-\rho^2)^{\frac{1}{2}}(1-z^2)^{\frac{3}{2}}(1-z)(1-2rz+z^2)^{\frac{n-5}{2}}}{(1-\rho)(1-2\rho r+\rho^2)^{\frac{n-3}{2}}(1-\rho z)(1-2\rho r+\rho z+\rho-z)}$$

$$\times [n-5 + \frac{2(1-z)(1-2\rho r+\rho^2)}{(1-\rho z)(1-2\rho r+\rho z+\rho-z)}] .$$

By equation (2.2.2) we have

$$h(r) \sim \frac{n(1-\rho^2)^{\frac{1}{2}}}{2\pi i\,(1-\rho)(1-2\rho r+\rho^2)^{\frac{n-3}{2}}} \int \varphi(z)(1-2rz+z^2)^{\frac{n-5}{2}} dz ,$$

where

$$(3.2.8) \quad \varphi(z) = \frac{(1-z^2)^{\frac{3}{2}}(1-z)}{(1-\rho z)(1-2\rho r+\rho z+\rho-z)} \{1 + \frac{1}{n} [\frac{2(1-z)(1-2\rho r+\rho^2)}{(1-\rho z)(1-2\rho r+\rho z+\rho-z)} - 5]\} .$$

The mapping and path of integration in the z-plane are the same as in the cases discussed previously. The only possible singularity in the integrand of the integral for $h(r)$ is real and occurs when

$$1 - 2\rho r + \rho z + \rho - z = 0 .$$

By confining z, on the real axis, to the following range, the singularity is avoided:

(i) $\rho \leq z \leq r$, $\rho < r$

(ii) $r \leq z \leq \rho$, $r < \rho$

(iii) $z = r = \rho$.

These cases are discussed in the previous section of this chapter and, in each case, it has been shown that

$$1 - 2\rho r + \rho z + \rho - z > 0 .$$

Hence, as before, we may deform the path of integration in the z-plane to be the straight line joining $e^{-i\theta}$ to $e^{i\theta}$, where $\cos \theta = r$, and cutting the real axis at $z = r$.

We may write z as given by equation (2.3.24) on the path of integration giving

$$h(r) \sim \frac{n(1-\rho^2)^{\frac{1}{2}}(1-r^2)^{\frac{n}{2}-2}}{2\pi\ (1-\rho)(1-2\rho r+\rho^2)^{\frac{n-3}{2}}} \int_{-1}^{1} \varphi(z)(1-w^2)^{\frac{n-5}{2}}\, dw .$$

Expanding $\varphi(z)$ in a Taylor's series about $z = r$ and using equation (2.3.24), it is seen that

$$\varphi(z) = \varphi(r) + iw(1-r^2)^{\frac{1}{2}}\varphi'(r) + \ldots + \frac{i^k w^k (1-r^2)^{\frac{k}{2}}}{k!}\varphi^{(k)}(r) + \ldots$$

and hence,

$$h(r) \sim \frac{n}{2\pi} \frac{(1-\rho^2)^{\frac{1}{2}}(1-r^2)^{\frac{n}{2}-2}}{(1-\rho)(1-2\rho r+\rho^2)^{\frac{n-3}{2}}} \sum_{k=0}^{\infty} \frac{i^k(1-r^2)^{\frac{k}{2}}}{k!}\varphi^{(k)}(r) \int_{-1}^{1} w^k (1-w^2)^{\frac{n-5}{2}} dw \ .$$

As before, if k is an odd integer, the integrand is an odd function and the integral vanishes, so that we write

$$h(r) \sim \frac{n}{2\pi} \frac{(1-\rho^2)^{\frac{1}{2}}(1-r^2)^{\frac{n}{2}-2}}{(1-\rho)(1-2\rho r+\rho^2)^{\frac{n-3}{2}}} \sum_{k=0}^{\infty} \frac{(-1)^k(1-r^2)^k}{(2k)!}\varphi^{(2k)}(r) \int_{-1}^{1} w^{2k}(1-w^2)^{\frac{n-5}{2}} dw \ .$$

To evaluate

$$I_k = \int_{-1}^{1} w^{2k} (1-w^2)^{\frac{n-5}{2}} dw \ ,$$

we follow the method used in Chapter II, section 3. Letting

$$w^2 = u$$

gives

$$I_k = \int_{0}^{1} u^{k-\frac{1}{2}} (1-u)^{\frac{n-5}{2}} du = \beta(k + \tfrac{1}{2}, \tfrac{n-3}{2})$$

$$= \frac{\Gamma(k + \frac{1}{2})\,\Gamma(\frac{n-3}{2})}{\Gamma(k + \frac{n}{2}-1)} \ .$$

Then

$$\text{for} \quad k = 0 \ , \quad I_0 = \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{n-3}{2})}{\Gamma(\frac{n}{2}-1)} = \frac{\sqrt{\pi}\,\Gamma(\frac{n-3}{2})}{\Gamma(\frac{n}{2}-1)} \ ;$$

$$k = 1 \ , \quad I_1 = \frac{\Gamma(\frac{3}{2})\,\Gamma(\frac{n-3}{2})}{\Gamma(\frac{n}{2})} = \frac{1}{n-2}\ I_0 \ ;$$

$$k = 2 \ , \quad I_2 = \frac{\Gamma(\frac{5}{2})\,\Gamma(\frac{n-3}{2})}{\Gamma(\frac{n}{2}+1)} = \frac{1.3}{n(n-2)}\ I_0 \ ;$$

and in general, for $k = 1, 2, \ldots$

$$(3.2.9) \qquad I_k = \frac{1.3 \ldots (2k-1)}{(n-2)n \ldots (n+2k-4)}\ I_0 \ ,$$

where

$$I_0 = \frac{\sqrt{\pi}\,\Gamma(\frac{n-3}{2})}{\Gamma(\frac{n}{2}-1)} \ .$$

Using equation (3.2.9) in the expression for $h(r)$, we have

$$h(r) \sim \frac{n}{2\pi} \frac{(1-\rho^2)^{\frac{1}{2}}(1-r^2)^{\frac{n}{2}-2}}{(1-\rho)(1-2\rho r+\rho^2)^{\frac{n-3}{2}}} I_0 \{\varphi(r) + \sum_{k=1}^{\infty} \frac{(-1)^k(1-r^2)^k}{(2k)!} \varphi^{(2k)}(r) \frac{[1.3\ldots(2k-1)]}{[(n-2)\ldots(n+2k-4)]}\}$$

or

$$h(r) \sim \frac{n}{2\sqrt{\pi}} \frac{\Gamma(\frac{n-3}{2})}{\Gamma(\frac{n}{2}-1)} \frac{(1-\rho^2)^{\frac{1}{2}}(1-r^2)^{\frac{n}{2}-2}}{(1-\rho)(1-2\rho r+\rho^2)^{\frac{n-3}{2}}} \{\varphi(r) + \sum_{k=1}^{\infty} \frac{(-1)^k(1-r^2)^k\,\varphi^{(2k)}(r)}{2^k k!\,[n-2)\ldots(n+2k-4)]}\} \ .$$

As mentioned in the previous cases, the dominant term in this asymptotic expansion is taken as the asymptotic value of $h(r)$. Then to $O(n^{-1})$ we have

$$(3.2.10)\quad h(r) \sim \frac{n}{2\sqrt{\pi}} \; \frac{\Gamma(\frac{n-3}{2})}{\Gamma(\frac{n}{2}-1)} \; \frac{(1-\rho^2)^{\frac{1}{2}}(1-r^2)^{\frac{n}{2}-2}}{(1-\rho)(1-2\rho r+\rho^2)^{\frac{n-3}{2}}} \; \varphi(r)\,[1 + O(n^{-1})] \quad .$$

Putting $z = r$ in equation (3.2.8) gives

$$\varphi(r) = \frac{(1-r^2)^{\frac{3}{2}}}{(1-\rho r)(1+\rho)} \; \{1 + \frac{1}{n}\,[\frac{2(1-2\rho r+\rho^2)}{(1-\rho r)(1+\rho)} - 5]\}$$

or

$$(3.2.11)\quad \varphi(r) = \frac{(1-r^2)^{\frac{3}{2}}}{(1-\rho r)(1+\rho)}\,[1 + O(n^{-1})] \quad .$$

On substituting equation (3.2.11) in equation (3.2.10) we obtain the following approximate value of $h(r)$:

$$h(r) \sim \frac{n}{2\sqrt{\pi}} \; \frac{\Gamma(\frac{n-3}{2})}{\Gamma(\frac{n}{2}-1)} \; \frac{(1-r^2)^{\frac{n-1}{2}}}{(1-\rho^2)^{\frac{1}{2}}(1-\rho r)(1-2\rho r+\rho^2)^{\frac{n-3}{2}}} \; [1 + O(n^{-1})]$$

since we neglect terms that are relatively $O(n^{-1})$.

By the method used in Chapter II, section 3, it is seen that $h(r)$ becomes

$$h(r) \sim K \; \frac{(1-r^2)^{\frac{N}{2}}}{(1-2\rho r+\rho^2)^{\frac{N-1}{2}}} \; [1 + O(n^{-\frac{3}{2}})] \quad ,$$

where K is an adjusted normalizing constant and

$$N = n - 1 + \frac{\rho^2}{1-\rho^2} \quad .$$

To renormalize $h(r)$, consider

$$I = \int_{-1}^{1} \frac{(1-r^2)^{\frac{N}{2}}}{(1-2\rho r+\rho^2)^{\frac{N-1}{2}}}\, dr$$

$$= \int_{-1}^{1} \frac{(1-r^2)^{\frac{N}{2}}(1-2\rho r+\rho^2)}{(1-2\rho r+\rho^2)^{\frac{N+1}{2}}}\, dr$$

$$= (1+\rho^2) \int_{-1}^{1} \frac{(1-r^2)^{\frac{N}{2}}}{(1-2\rho r+\rho^2)^{\frac{N+1}{2}}}\, dr - 2\rho \int_{-1}^{1} \frac{r(1-r^2)^{\frac{N}{2}}}{(1-2\rho r+\rho^2)^{\frac{N+1}{2}}}\, dr \quad .$$

With $k = N+1$ in equation (2.4.4), we see that

$$\int_{-1}^{1} \frac{(1-r^2)^{\frac{N}{2}}}{(1-2\rho r+\rho^2)^{\frac{N+1}{2}}}\, dr = \frac{\sqrt{\pi}\ \Gamma(\frac{N}{2}+1)}{\Gamma(\frac{N+3}{2})} \quad .$$

Integrating by parts and using equation (2.4.4) with $k = N+3$ gives

$$\int_{-1}^{1} \frac{r(1-r^2)^{\frac{N}{2}}}{(1-2\rho r+\rho^2)^{\frac{N+1}{2}}}\, dr = \frac{\rho(N+1)}{(N+2)} \int_{-1}^{1} \frac{(1-r^2)^{\frac{N}{2}+1}}{(1-2\rho r+\rho^2)^{\frac{N+3}{2}}}\, dr$$

$$= \frac{\rho\sqrt{\pi}\ (N+1)}{(N+2)} \frac{\Gamma(\frac{N}{2}+2)}{\Gamma(\frac{N+5}{2})} \quad .$$

Then

$$I = (1+\rho^2) \left[\frac{\sqrt{\pi}\ \Gamma(\frac{N}{2}+1)}{\Gamma(\frac{N+3}{2})} \right] - 2\rho \left[\frac{\rho\sqrt{\pi}\ (N+1)}{(N+2)} \frac{\Gamma(\frac{N}{2}+2)}{\Gamma(\frac{N+5}{2})} \right]$$

$$= \frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{N}{2}+1)}{\Gamma(\frac{N+5}{2})} [(1-\rho^2)(N-1) + 4] \quad .$$

Thus, the renormalized density function, $h(r)$, in Leipnik's form is

$$h(r) \sim K(n,\rho) \frac{(1-r^2)^{\frac{N}{2}}}{(1-2\rho r+\rho^2)^{\frac{N-1}{2}}} [1 + O(n^{-\frac{3}{2}})] \quad , \tag{3.2.12}$$

where

$$K(n,\rho) = \frac{1}{I} = \frac{2}{\sqrt{\pi}} \quad \frac{\Gamma(\frac{N+5}{2})}{\Gamma(\frac{N}{2}+1)[1-\rho^2)(N-1)+4]}$$

and

$$N = n-1 + \frac{\rho^2}{1-\rho^2} \quad .$$

CHAPTER IV

THE APPROXIMATE DISTRIBUTION OF A RATIO OF THE SQUARE SUCCESSIVE DIFFERENCE TO THE SQUARE DIFFERENCE

We are interested in obtaining the approximate distributions of the statistics

$$D = \frac{\sum_{i=1}^{n-1} (x_{i+1} - x_i)^2}{\sum_{i=1}^{n} x_i^2}$$

and

$$D_1 = \frac{\sum_{i=1}^{n-1} (x_{i+1} - x_i)^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2} ,$$

where

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} .$$

As mentioned in Chapter I these statistics are similar to

$$\eta = \frac{\delta^2}{s^2} ,$$

where

$$\delta^2 = \frac{\sum_{i=1}^{n-1} (x_{i+1} - x_i)^2}{n-1}$$

and

$$s^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n} ,$$

when n is large. D is the case where the mean is known, and D_1 is

the one where it is unknown but estimated by $\bar{x}$. We assume that the sample consists of serially correlated rather than independent, normal random variables.

Let us first consider the distribution of D. We see that

$$r = 1 - \frac{1}{2} D ,$$

where

$$r = \frac{c}{c_0}$$

with c, c_0 given by (3.1.1). Then

$$g(D) = \frac{1}{2} h(r) \Big|_{r = 1 - \frac{1}{2} D}$$

so that, using equation (3.1.17),

$$g(D) \sim \frac{1}{2} K(n,\rho) \frac{D^{\frac{N-1}{2}} (4-D)^{\frac{N+1}{2}}}{2^N [(1-\rho)^2 + \rho D]^{\frac{N}{2}}} [1 + O(n^{-\frac{3}{2}})]$$

or

$$g(D) \sim K_1(n,\rho) \, D^{\frac{N-1}{2}} (4-D)^{\frac{N+1}{2}} [1 + \frac{\rho D}{(1-\rho)^2}]^{-\frac{N}{2}} [1 + O(n^{-\frac{3}{2}})] ,$$

where

$$K_1(n,\rho) = \frac{\Gamma(\frac{N}{2} + 2)}{2^N \sqrt{\pi} \, (1-\rho)^N \, \Gamma(\frac{N+1}{2})[N(1+\rho)+2]}$$

and

$$N = n - 1 + \frac{\rho^2}{1-\rho^2} .$$

As r ranges from -1 to 1, D ranges from 0 to 4.

To obtain the distribution of D_1, we use the density function $h(r)$ as given by equation (3.2.12), since

$$r = 1 - \frac{1}{2} D_1 ,$$

where

$$r = \frac{C}{C_0}$$

with C, C_0 given by (3.2.2). Hence,

$$g(D_1) = \frac{1}{2} h(r) \Big|_{r = 1 - \frac{1}{2} D_1}$$

giving

$$g(D_1) \sim \frac{1}{2} K(n,\rho) \frac{D_1^{\frac{N}{2}} (4-D_1)^{\frac{N}{2}}}{2^N [(1-\rho)^2+\rho D_1]^{\frac{N-1}{2}}} [1 + O(n^{-\frac{3}{2}})]$$

or

$$g(D_1) \sim K_2(n,\rho) D_1^{\frac{N}{2}} (4-D_1)^{\frac{N}{2}} [1 + \frac{\rho D_1}{(1-\rho)^2}]^{-(\frac{N-1}{2})} [1 + O(n^{-\frac{3}{2}})] ,$$

where

$$K_2(n,\rho) = \frac{\Gamma(\frac{N+5}{2})}{2^N \sqrt{\pi} (1-\rho)^{N-1} \Gamma(\frac{N}{2}+1)[(N-1)(1-\rho^2)+4]}$$

and

$$N = n - 1 + \frac{\rho^2}{1-\rho^2} .$$

Here also, D_1 ranges from 0 to 4 as r ranges from -1 to 1.

Now, we shall show that the moments of

$$\frac{A}{B} = \frac{\sum_{j=1}^{n-1} (X_{j+1} - X_j)^2}{\sum_{j=1}^{n} (X_j - \bar{X})^2} ,$$

and the mean value and variance of

$$\eta = \frac{\delta^2}{s^2} = \frac{n}{n-1} \frac{A}{B} ,$$

obtained by Williams (16), are approximately the same as those we obtain from the density function of D_1 with $\rho = 0$. Let m_k denote the k^{th} moment about the origin. Since equation (3.2.12) is the renormalized density function of r and that of D_1 is obtained from it,

$$m_0 = 1 .$$

Using the density function of D_1 with $\rho = 0$, we see that

$$m_1 = K_2(n,0) \int_0^4 D_1^{\frac{n+1}{2}} (4-D_1)^{\frac{n-1}{2}} dD_1 ,$$

where

$$K_2(n,0) = \frac{\Gamma(\frac{n}{2}+2)}{2^{n-1} \sqrt{\pi} (n+2) \Gamma(\frac{n+1}{2})} .$$

Putting $D_1 = 4x$ gives

$$m_1 = \frac{2^{n+3} \Gamma(\frac{n}{2}+2)}{\sqrt{\pi} (n+2)\Gamma(\frac{n+1}{2})} \int_0^1 x^{\frac{n+1}{2}} (1-x)^{\frac{n-1}{2}} dx$$

$$= \frac{2^{n+3}\,\Gamma(\frac{n}{2}+2)}{\sqrt{\pi}\,(n+2)\Gamma(\frac{n+1}{2})} \cdot \beta(\frac{n+3}{2}, \frac{n+1}{2})$$

$$= \frac{2^{n+3}\,\Gamma(\frac{n}{2}+2)\,\Gamma(\frac{n+3}{2})}{\sqrt{\pi}\,\Gamma(n+3)} \quad .$$

By [(15), page 240],

$$m_1 = \frac{2^{n+3}\,\Gamma(\frac{n}{2}+2)\,\Gamma(\frac{n+3}{2})}{2^{n+2}\,\Gamma(\frac{n}{2}+2)\,\Gamma(\frac{n+3}{2})} = 2 \quad .$$

Similarly,

$$m_2 = \frac{4(n+3)}{n+2} \quad \text{and} \quad m_3 = \frac{8(n+5)}{n+2} \quad .$$

Now, m_0 and m_1 are the same as the corresponding moments obtained by Williams [(16), (10)]. Also,

$$m_2 = 4[\frac{n+3}{n+2}] = 4[1 + \frac{1}{n} - \frac{2}{n^2} + O(n^{-3})] \quad ,$$

and since Williams' value is

$$m_2 = 4[\frac{n^2+n-3}{(n+1)(n-1)}] = 4[1 + \frac{1}{n} - \frac{2}{n^2} + O(n^{-3})] \quad ,$$

the second moments about the origin are the same to $O(n^{-3})$. In a similar manner, we see that the third moments about the origin are the same to $O(n^{-3})$. Hence, for large values of n these moments of D_1 with $\rho = 0$ are similar to those of $\frac{A}{B}$ as obtained by Williams. We can show that this is also the case for higher moments as well.

Considering the moments m_1 and m_2 of D_1 with $\rho = 0$, as obtained above, we see that the mean value is 2 and the variance is

$$\text{Variance} = m_2 - (m_1)^2$$
$$= \frac{4}{n+2} \quad .$$

The mean value of η is

$$\frac{2n}{n-1} \quad .$$

In his paper Williams, [(16), page 241], has given an incorrect value for the variance of η. Using the moments m_1 and m_2 of $\frac{A}{B}$ obtained by Williams, we see that the correct value of the variance of η is

$$\frac{4n^2(n-2)}{(n+1)(n-1)^3} \quad .$$

For large values of n the mean of η is approximately 2 and therefore similar to our value. Since

$$\frac{4n^2(n-2)}{(n+1)(n-1)^3} = 4[\frac{1}{n} + O(n^{-4})]$$

and

$$\frac{4}{n+2} = 4[\frac{1}{n} + O(n^{-2})] \quad ,$$

for large values of n the value of the variance that we obtained and the one obtained by Williams are approximately equal. Thus we see that for large values of n the distribution of D_1 with $\rho = 0$ is approximately equal to that of η discussed by von Neumann (10), (11) and Williams (16).

APPENDIX I

EVALUATION OF A DETERMINANT WHICH IS FREQUENTLY ENCOUNTERED IN STATISTICS

We are interested in finding the value of a determinant of the form

$$(\text{I.1}) \qquad A_n = \begin{vmatrix} f & b & & & & & \\ b & a & b & & & & \\ & b & a & b & & (c) & \\ & & \cdot & \cdot & \cdot & & \\ & (c) & & \cdot & \cdot & \cdot & \\ & & & & & a & b \\ & & & & & b & f \end{vmatrix}_n .$$

The technique of evaluating determinants by difference equations, used here, is due to Dixon (3).

Replacing f by $[a + (f-a)]$ in (I.1), we obtain

$$(\text{I.2}) \qquad A_n = E_n + 2(f-a)\, E_{n-1} + (f-a)^2\, E_{n-2} ,$$

where

$$E_n = \begin{vmatrix} a & b & & & & & \\ b & a & b & & & (c) & \\ & b & a & b & & & \\ & & \cdot & \cdot & \cdot & & \\ & (c) & & \cdot & \cdot & \cdot & \\ & & & & & a & b \\ & & & & & b & a \end{vmatrix}_n .$$

The determinant above, denoted by E_n, is the same as E_n of (4.10)

of Dixon's paper (3). Equation (4.2.3) of (3) is

$$(\mathrm{I}.3) \qquad E_n = H_n - c\, I_{n+1} \quad ,$$

where

$$H_n = \begin{vmatrix} s & r & & & & & \\ r & s & r & & & & \\ & r & s & r & & (0) & \\ & & \cdot & \cdot & \cdot & & \\ (0) & & & \cdot & \cdot & \cdot & \\ & & & & & s & r \\ & & & & & r & s \end{vmatrix}_n \quad ,$$

$$r = b-c \quad , \quad s = a-c \quad ,$$

and

$$I_n = \begin{vmatrix} 0 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & 1 & 1 \\ 1 & s & r & & & & & & \\ 1 & r & s & r & & & & (0) & \\ \cdot & & r & s & r & & & & \\ \cdot & & & \cdot & \cdot & \cdot & & & \\ \cdot & (0) & & & \cdot & \cdot & \cdot & & \\ 1 & & & & & & & s & r \\ 1 & & & & & & & r & s \end{vmatrix}_n \quad .$$

As shown in (3), the value of H_n is

$$(\mathrm{I}.4) \qquad H_n = \frac{u^{n+1} - v^{n+1}}{u-v} \quad ,$$

where u and v are the roots of the equation

$$x^2 - sx + r^2 = 0 \quad .$$

Then

$$(I.5)\qquad u + v = s$$

and

$$(I.6)\qquad uv = r^2 .$$

The value of I_n is obtained by solving the difference equation

$$(I.7)\qquad I_n - s\, I_{n-1} + r^2\, I_{n-2} = (-1)^{n-1}(G_{n-1} + r\, G_{n-2}) ,$$ [(3), equation (4.27)],

where the value of

$$G_n = \begin{vmatrix} 1 & s & r & & & & & \\ 1 & r & s & r & & & & \\ 1 & & r & s & r & & (0) & \\ \cdot & & & \cdot & \cdot & \cdot & & \\ \cdot & (0) & & & \cdot & \cdot & \cdot & \\ 1 & & & & & & r & s \\ 1 & & & & & & & r \end{vmatrix}_n ,$$ [(3), (4.18)] ,

is

$$(I.8)\qquad G_n = \frac{r^n}{2r+s} + (-1)^{n-1} \left[\frac{r(u^n - v^n) + u^{n+1} - v^{n+1}}{(u-v)(2r+s)} \right] ,$$ [(3), equation (4.20)] .

Equation (I.7) is equivalent to

$$(I.9)\qquad I_{n+2} - s\, I_{n+1} + r^2\, I_n = (-1)^{n-1}\, (G_{n+1} + r\, G_n).$$

Letting E be the forward difference operator of the calculus of finite differences as defined by (2.3.8), equation (I.9) may be written as

$$(E^2 - sE + r^2)\, I_n = (-1)^{n-1} (G_{n+1} + r\, G_n) .$$

Now, using equations (I.5), (I.6) and (I.8), we have

$$G_{n+1} + r\, G_n = \{\frac{r^{n+1}}{2r+s} + (-1)^n [\frac{r(u^{n+1}-v^{n+1})+u^{n+2}-v^{n+2}}{(u-v)(2r+s)}]\}$$

$$+ r\, \{\frac{r^n}{2r+s} + (-1)^{n-1} [\frac{r(u^n-v^n)+u^{n+1}-v^{n+1}}{(u-v)(2r+s)}]\}$$

$$= \frac{2r^{n+1} + (-1)^n (u^{n+1} + v^{n+1})}{2r+s} .$$

Hence, we wish to solve the difference equation

$$\text{(I.10)} \qquad (E^2-sE+r^2)\, I_n = \frac{2(-r)^{n+1} - (u^{n+1} + v^{n+1})}{2r+s} .$$

To obtain the complementary function of equation (I.10) we consider

$$(E^2 - s\, E + r^2)\, I_n = 0 .$$

The corresponding auxiliary equation,

$$x^2 - sx + r^2 = 0 ,$$

has roots u and v. Then

$$I_n = K_1\, u^n + K_2\, v^n$$

is the complementary function of the complete solution of equation (I.10).

The particular integral is obtained as follows: We assume a solution of the form

$$(I.11) \qquad I_n = A(-r)^{n+1} + Bnu^n + Cnv^n \quad .$$

Using equations (I.5), (I.6) and (I.11) in equation (I.10), we have

$$(E^2-sE+r^2)\, I_n = -A(2r+s)(-r)^{n+2} + B(u-v)u^{n+1} - C(u-v)v^{n+1}$$

$$= \frac{2(-r)^{n+1} - (u^{n+1} + v^{n+1})}{2r+s} \quad .$$

Equating coefficients gives us

$$A = \frac{2}{r(2r+s)^2} \quad , \quad B = \frac{-1}{(2r+s)(u-v)} \quad , \quad C = \frac{1}{(2r+s)(u-v)} \quad .$$

Then equation (I.11) becomes

$$I_n = \frac{2(-1)^{n-1}\, r^n}{(2r+s)^2} - n\left[\frac{u^n - v^n}{(u-v)(2r+s)}\right] \quad ,$$

and the complete solution of equation (I.10) is

$$(I.12) \qquad I_n = K_1\, u^n + K_2\, v^n + \frac{2(-1)^{n-1} r^n}{(2r+s)^2} - n\left[\frac{u^n - v^n}{(u-v)(2r+s)}\right] \quad .$$

Knowing that

$$I_1 = 0$$

and

$$I_2 = -1 \quad ,$$

and using equations (I.5) and (I.6), we can solve for K_1 and K_2 in equation (I.12). Thus

$$(I.13)\quad \begin{cases} I_1 = 0 = K_1 u + K_2 v - \dfrac{s}{(2r+s)^2} \quad , \\[2ex] I_2 = -1 = K_1 u^2 + K_2 v^2 - \dfrac{(u^2+v^2)}{(2r+s)^2} \quad . \end{cases}$$

Solving (I.13) for K_1 and K_2, we obtain

$$K_1 = \frac{1}{(2r+s)^2} \quad , \quad K_2 = \frac{1}{(2r+s)^2} \quad ,$$

so that equation (I.12) becomes

$$(I.14)\qquad I_n = \frac{u^n+v^n+2(-1)^{n-1}r^n}{(2r+s)^2} - n\left[\frac{u^n-v^n}{(u-v)(2r+s)}\right] \quad .$$

Using equations (I.4) and (I.14) in equation (I.3), the value of E_n is found to be

$$E_n = \frac{u^{n+1}-v^{n+1}}{u-v} - c\left\{\frac{u^{n+1}+v^{n+1}+2(-1)^n r^{n+1}}{(2r+s)^2} - (n+1)\left[\frac{u^{n+1}-v^{n+1}}{(u-v)(2r+s)}\right]\right\}$$

$$= \frac{u^{n+1}}{(u-v)(2r+s)^2}\{(2r+s)^2-c[(u-v)-(n+1)(2r+s)]\}$$

$$- \frac{v^{n+1}}{(u-v)(2r+s)^2}\{(2r+s)^2+c[(u-v)+(n+1)(2r+s)]\}$$

$$+ \frac{2\ c(-r)^{n+1}}{(2r+s)^2} \quad .$$

Substituting this value of E_n in equation (I.2) and collecting terms, we have

$$(I.15)\quad A_n = \frac{u^{n-1}}{(u-v)(2r+s)^2}\{[u+(f-a)]^2[(2r+s)^2-c((u-v)-n(2r+s))] + c(2r+s)[u^2-(f-a)^2]\}$$

$$- \frac{v^{n-1}}{(u-v)(2r+s)^2}\{[v+(f-a)]^2[(2r+s)^2+c((u-v)+n(2r+s))] + c(2r+s)[v^2-(f-a)^2]\}$$

$$+ \frac{2c(\ -r)^{n-1}}{(2r+s)^2}[r-(f-a)]^2 \quad ,$$

where u and v are the roots of

$$x^2 - sx + r^2 = 0$$

and

$$r = b-c, \quad s = a-c \quad .$$

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